

# Towards pure spinor type covariant description of supermembrane — an approach from the double spinor formalism

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**ABSTRACT:** In a previous work, we have constructed a reparametrization invariant world-sheet action from which one can derive the super-Poincaré covariant pure spinor formalism for the superstring at the fully quantum level. The main idea was the doubling of the spinor degrees of freedom in the Green-Schwarz formulation together with the introduction of a new compensating local fermionic symmetry. In this paper, we extend this “double spinor” formalism to the case of the supermembrane in 11 dimensions at the classical level. The basic scheme works in parallel with the string case and we are able to construct the closed algebra of first class constraints which governs the entire dynamics of the system. A notable difference from the string case is that this algebra is first order reducible and the associated BRST operator must be constructed accordingly. The remaining problems which need to be solved for the quantization will also be discussed.

**KEYWORDS:** Superspaces, p-branes, Superstrings and Heterotic Strings, BRST Symmetry.

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## 1. Introduction

Six years ago, N. Berkovits opened up a novel perspective for the quantization of the superstring with manifest super-Poincaré covariance by proposing the so-called pure spinor (PS) formalism [1]. The basic ingredient of this formalism is the BRST-like operator  $Q = \int dz \lambda^\alpha d_\alpha$ , where  $d_\alpha = p_\alpha + i\partial x^m (\gamma_m \theta)_\alpha + \frac{1}{2} (\gamma^m \theta)_\alpha (\theta \gamma^m \partial \theta)$  coincides with the familiar

constraint that arises in the conventional Green-Schwartz (GS) formalism and  $\lambda^\alpha$  is a bosonic chiral spinor playing the role of the associated “ghost”. For  $Q$  to be regarded as a BRST operator, however,  $\lambda^\alpha$  must satisfy a subsidiary constraint. With the assumption that all the fields are free, one obtains the operator product  $d_\alpha(z)d_\beta(w) = 2i\gamma_{\alpha\beta}^m(\partial x_m - i\theta\gamma_m\partial\theta)/(z-w)$  and hence  $Q$  becomes nilpotent if and only if  $\lambda^\alpha\gamma_{\alpha\beta}^m\lambda^\beta = 0$ . This, in 10 dimensions, is precisely the condition for  $\lambda^\alpha$  to be a *pure spinor* in the sense of Cartan [2].

The striking property of this operator  $Q$  is that, despite its simplicity, its cohomology correctly reproduces the spectrum of the superstring [3]. Moreover, together with the field  $\omega_\alpha$  conjugate to  $\lambda^\alpha$ , the fields in the theory form a conformal field theory (CFT) with vanishing central charge, which allows one to make use of the powerful machinery of CFT.  $Q$ -invariant vertex operators were constructed and by postulating an appropriate functional measure the known tree level amplitudes were reproduced in a manifestly covariant manner [1, 4]. Subsequently, this success was extended to the multi-loop level [5]. Some explicit supercovariant calculations have been performed at 1 and 2 loops [6], which agreed with the results obtained in the Ramond-Neveu-Schwarz (RNS) formalism [7, 8]. Furthermore certain vanishing theorems were proved to all orders for the first time, demonstrating the power of this formalism [5]. Another advantage of the PS formalism is that it can be coupled to backgrounds including Ramond-Ramond fields in a quantizable and covariant way [1, 9], in distinction to the conventional RNS and GS formalisms, where one encounters difficulties.

More recently it has been shown that, with some additional fields, the original PS formalism can be promoted to a new “topological” formulation [10]<sup>1</sup>, where the structure of the loop amplitude becomes very similar to the bosonic string, just like in the case of the topological string [12]. This structure may shed more light on the deeper understanding of the PS formalism. For many other developments, the reader is referred to [14]–[28] and a review [29].

Behind these remarkable advances, there remained a number of important mysteries concerning this formalism: What is the underlying reparametrization invariant worldsheet action and what are its symmetries? How does  $Q$  arise as a BRST operator? Why are all the fields free? How does  $\lambda^\alpha$  get constrained and how does one quantize it? Why is the Virasoro constraint absent in  $Q$ ? How does one derive the functional measure? In summary, the basic problem was to understand the origin of the PS formalism.

In a previous work [22], we have given answers to many of the above questions by constructing a fundamental reparametrization invariant action from which one can derive the PS formalism at the fully quantum level. As we shall review in section 2, the basic idea was to add a new spinor degree of freedom  $\theta^\alpha$  to the Green-Schwarz action consisting of  $x^m$  and  $\tilde{\theta}^\alpha$ , in such a way that a compensating local fermionic symmetry appears on top of the usual  $\kappa$ -symmetry [30]. Due to this extra symmetry, the physical degrees of freedom remain unchanged. Just as in the usual GS formalism, the standard Hamiltonian analysis à la Dirac shows that both the first and the second class constraints arise, which cannot be separated without breaking manifest Lorentz covariance. Now the advantage

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<sup>1</sup>See also [11].

of the “double spinor” formalism is that this breakdown can be confined to the  $\tilde{\theta}$  sector while the covariance for  $x^m$  and  $\theta^\alpha$  remain intact. Then after fixing the  $\kappa$ -symmetry for the  $\tilde{\theta}$  sector by adopting the semi-light-cone (SLC) gauge, one obtains a closed set of first class constraints which govern the entire dynamics of the theory. This algebra, which is absent in the conventional GS formalism, has its origin in the aforementioned extra local fermionic symmetry and is the most important feature of the double spinor formalism.

Furthermore, by appropriate redefinitions of the momenta, one can construct a basis of fields in which the Dirac brackets among them take the canonical “free field” form. This at the same time simplifies the form of the constraints. The quantization can then be performed by replacing the Dirac bracket by the quantum bracket together with slight quantum modifications of the form of the constraints due to multiple contractions and normal-ordering. The quantum first class algebra so obtained precisely matches the one proposed in [17] and justifies the free-field postulate of Berkovits.<sup>2</sup>

The nilpotent BRST operator  $\hat{Q}$  associated with it can easily be constructed in the standard way, with the introduction of *unconstrained* bosonic spinor ghosts  $\tilde{\lambda}^\alpha$  and the reparametrization ghosts  $(b, c)$  associated with the Virasoro constraint. At this stage,  $\hat{Q}$  still contains non-covariant pieces representing the part of the degrees of freedom of the gauge-fixed  $\tilde{\theta}^\alpha$ . The remarkable fact is that all the unwanted components in  $\hat{Q}$  can be removed or cohomologically decoupled through a quantum similarity transformation: The Virasoro generator disappears together with the  $b, c$  ghosts and the non-covariant remnants of  $\tilde{\theta}^\alpha$  cancel against a part of the unconstrained  $\tilde{\lambda}^\alpha$  in such a way that it precisely becomes a pure spinor  $\lambda^\alpha$  satisfying the quadratic PS constraint. In this way one finally arrives at the Berkovits’ expression  $Q = \int dz \lambda^\alpha d_\alpha$ . In [22] it was also shown that the same method can be used to derive the PS formalism for a superparticle in 11 dimensions.

Now, an obvious and challenging question arises: Is the above idea applicable to the supermembrane in 11 dimensions as well?

Some years ago, the possibility of a pure spinor type formalism for the supermembrane was investigated by Berkovits [16]. Largely based on the requirement that the theory should reduce in appropriate limits to that of a 11 dimensional superparticle and a 10 dimensional type IIA superstring, he generalized the conventional supermembrane action first written down by Bergshoeff, Sezgin and Townsend (BST) [31] to include a bosonic spinor  $\lambda^\alpha$  and its conjugate  $\omega_\alpha$ . This action is invariant under a postulated BRST transformation generated by  $Q = \int d^2\sigma \lambda^\alpha d_\alpha$ , which is nilpotent if a set of constraints on  $\lambda^\alpha$  are satisfied. In addition to the familiar one  $\bar{\lambda}\Gamma^M\lambda = 0$ , this set includes further new constraints involving worldvolume derivatives. Unlike the case of the superparticle and the superstring, the action is non-linear and the problem of quantization was left unsolved. Nonetheless, this pioneering study gave some hope that a covariant quantization of a supermembrane may be possible along the lines of the pure spinor formalism. Our work to be presented in this paper is another attempt for this challenging task from a different more systematic point of view.

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<sup>2</sup>This first class algebra can also be obtained by the so-called “BFT embedding method” [13]

We will now outline the results of our investigation, which at the same time indicates the organization of the paper.

We begin in section 2 with a review of how the double spinor formalism works in the case of the superparticle and the superstring. This should help the reader to form a clear picture of the basic mechanism, without being hampered by the complicated details of the membrane case stemming from the high degree of added non-linearity.

The main analysis for the supermembrane case is performed in section 3. The fundamental action we start with in section 3.1 is formally of the same form as the conventional one [31], except that (i) the spinor variable  $\tilde{\theta}_A$  is replaced by  $\tilde{\theta}_A - \theta_A$  where  $\theta_A$  is the newly introduced spinor and (ii) the membrane coordinate  $x^M$  is replaced by  $x^M - i\bar{\theta}\Gamma^M\tilde{\theta}$ . Due to these modifications, the action acquires an extra local fermionic symmetry, which will play the crucial role. Then, through the usual Dirac analysis, we obtain in section 3.2 the fundamental constraints of the system. Due to the presence of the extra spinor  $\theta_A$  and its conjugate momentum, there will be an additional fermionic constraint  $D_A$  besides the usual one  $\tilde{D}_A$  associated with  $\tilde{\theta}_A$ . Upon defining the Poisson brackets for the fundamental fields, we compute the algebra of constraints. This reveals, just as in the case of the superstring, a half of  $\tilde{D}_A$  are second class and the remaining half are first class. On the other hand, the combination  $\Delta_A = D_A + \tilde{D}_A$ , which generates the extra fermionic symmetry, anticommutes with both  $D_A$  and  $\tilde{D}_A$ . To separate the first and the second class part of  $\tilde{D}_A$ , we will make use of the light-cone decomposition. Then the first class part  $\tilde{K}_{\dot{\alpha}}$  can be identified as the generator of the  $\kappa$  transformation. Although the computations are much more involved compared to the string case, we show that the anticommutator  $\{\tilde{K}_{\dot{\alpha}}, \tilde{K}_{\dot{\beta}}\}$  closes into a bosonic expression  $\mathcal{T}_{\dot{\alpha}\dot{\beta}}$ , which, although somewhat complicated, is equivalent to the bosonic constraints coming from the original worldvolume reparametrization invariance. Next in section 3.3, we perform the semi-light-cone gauge fixing and eliminate the  $\kappa$ -generators as well as the original second class constraints, by defining the appropriate Dirac brackets. We are then left with the remaining fermionic constraints  $D_A$  and the bosonic constraints  $\mathcal{T}_{\dot{\alpha}\dot{\beta}}$ . Direct computation of the algebra of these quantities under the Dirac bracket is unwieldy but we found a way to determine it efficiently by indirect means. The result is a conceptually simple first class algebra, which governs the entire dynamics of the system. A notable difference from the string case, however, is that this algebra is first order reducible, namely that there is a linear relation between some of the constraints. The associated BRST operator, therefore, must be constructed according to the general theory [35] applicable to such a situation. This procedure is described in section 3.4.

Thus, as far as the general scheme of the double spinor formalism is concerned, we have found that it is indeed applicable to the supermembrane case as well and produces a BRST operator in which the covariance is retained for the bosonic coordinate  $x^M$  and the new spinor  $\theta$ , which is crucial for the would-be pure spinor type covariant formulation. Unfortunately, the remaining steps for proper quantization and elimination of the non-covariant remnants present a number of difficulties and at the present stage we have not yet been able to obtain complete solutions.

In preparation for future developments, however, we will spell out in section 4 the nature of these problems and present some preliminary investigations. The first problem,

discussed in section 4.1, is the construction of the basis in which the fields become “free” under the Dirac bracket. In the case of the superstring, this problem was solved completely in closed form, which was a crucial ingredient for the justification of the free field postulate of Berkovits. For the supermembrane, it gets considerably more complicated. Nevertheless, we will show that the desired basis can be explicitly constructed in closed form in the case of the usual BST formulation without the new spinor  $\theta$ . For the double spinor formalism, it is accomplished as yet partially but the result strongly indicates the existence of such a basis. The second problem is that of quantization. Even if such a “free field” basis is found, the replacement of the Dirac bracket by the quantum bracket is but a part of the quantization procedure. We will discuss what should be achieved for a complete quantization and present a preliminary analysis.

Finally, in section 5 we will briefly summarize the main points of this investigation and discuss future problems.

Four appendices are provided: Our notations and conventions are summarized in appendix A, useful formulas in the SLC gauge are collected in appendix B, a proof of the equivalence of sets of bosonic constraints is given in appendix C and the first order reducibility function is obtained in appendix D.

## 2. Basic idea of the double spinor formalism: a review

Let us begin with a review of the double spinor formalism for the case of the lower dimensional objects, which should serve as a reference point for the more complicated supermembrane case. To highlight the essence of the basic idea, we will concentrate on the simpler case of the superparticle and then supplement some further technical refinements needed for the superstring.

### 2.1 Superparticle

To motivate the double spinor formalism, it is useful to first recall the origin of the difficulty of the covariant quantization of a superparticle in the conventional Brink-Schwarz (BS) formalism [32]. In this formulation, a (type I) superparticle in 10 dimensions is described by the reparametrization invariant action given by

$$S_{BS} = \int dt \frac{1}{2e} \Pi^m \Pi_m, \quad \Pi^m = \dot{x}^m - i\tilde{\theta} \gamma^m \dot{\tilde{\theta}},$$

where  $e$  is the einbein,  $\tilde{\theta}^\alpha$  is a 16 dimensional Majorana-Weyl spinor, and the Lorentz vector index  $m$  runs from 0 to 9. The generalized momentum  $\Pi^m$ , and hence the action, is invariant under the global supersymmetry transformation  $\delta\tilde{\theta}^\alpha = \epsilon^\alpha, \delta x^m = i\epsilon \gamma^m \tilde{\theta}$ . In addition, the action is invariant under the  $\kappa$ -symmetry transformation of the form [30]  $\delta\tilde{\theta} = \Pi_m \gamma^m \kappa, \delta x^m = i\tilde{\theta} \gamma^m \delta\tilde{\theta}, \delta e = 4ie\dot{\tilde{\theta}}\kappa$ , where  $\kappa_\alpha$  is a local fermionic parameter.

From the definitions of the momenta  $(p_m, \tilde{p}_\alpha, p_e)$  conjugate to  $(x^m, \tilde{\theta}^\alpha, e)$  respectively, one obtains the following two primary constraints:

$$\tilde{d}_\alpha \equiv \tilde{p}_\alpha - ip_m (\gamma^m \tilde{\theta})_\alpha = 0, \quad p_e = 0. \tag{2.1}$$

Then, the consistency under the time development generated by the canonical Hamiltonian  $H = (e/2)p^2$  requires the additional bosonic constraint

$$T \equiv \frac{1}{2}p^2 = 0. \quad (2.2)$$

Hereafter, we will drop  $(e, p_e)$  by choosing the gauge  $e = 1$ . Then, taking the basic Poisson brackets as

$$\{x^m, p_n\}_P = \delta_n^m, \quad \{\tilde{p}_\alpha, \tilde{\theta}^\beta\}_P = -\delta_\alpha^\beta, \quad (2.3)$$

the remaining constraints form the algebra

$$\{\tilde{d}_\alpha, \tilde{d}_\beta\}_P = 2i\rlap{/}p_{\alpha\beta}, \quad \{\tilde{d}_\alpha, T\}_P = \{T, T\}_P = 0. \quad (2.4)$$

This is where the necessity of non-covariant treatment becomes evident: On the constrained surface  $p^2 = 0$ , the quantity  $\rlap{/}p$  has rank 8, indicating that eight of the  $\tilde{d}_\alpha$  are of second class and the remaining eight are of first class. Since there is no eight-dimensional representation of the Lorentz group, manifest covariance must be sacrificed in order to separate these two types of constraints.

To perform the separation, one employs the  $SO(8)$  decomposition

$$p^m = (p^+, p^-, p^i), \quad p^\pm = p^0 \pm p^9, \quad i = 1 \sim 8, \quad (2.5)$$

$$\tilde{d}_\alpha = (\tilde{d}_a, \tilde{d}_{\dot{a}}), \quad a, \dot{a} = 1 \sim 8. \quad (2.6)$$

Then from (2.4) it is easily checked that  $\tilde{d}_a$  are of second class and the combinations

$$\tilde{K}_{\dot{a}} \equiv \tilde{d}_{\dot{a}} - \frac{p_i}{p^+} \gamma_{\dot{a}b}^i \tilde{d}_b, \quad (2.7)$$

which generate the  $\kappa$ -transformation, form a set of first class constraints together with  $T$ . The rest of the procedure is standard: The second class constraints are handled by introducing the Dirac bracket, while the first class constraints can be treated by adopting the (semi-)light-cone gauge. After that the quantization can be performed in a straightforward manner.

The well-known analysis recalled above clearly shows that, as long as one employs a single spinor  $\tilde{\theta}^\alpha$ , there is no way to *derive* a covariant quantization scheme such as the pure spinor formalism of our interest. As we demonstrated in a previous work, this problem can be overcome by introducing an additional spinor  $\theta^\alpha$ , together with a new compensating local fermionic symmetry to keep the physical content of the theory intact.

The new action is formally the same as the Brink-Schwarz action (2.1), except that  $\tilde{\theta}$  and  $x^m$  are replaced as

$$\tilde{\theta} \rightarrow \Theta \equiv \tilde{\theta} - \theta, \quad x^m \rightarrow y^m \equiv x^m - i\theta\gamma^m\tilde{\theta}. \quad (2.8)$$

As we keep the new spinor  $\theta$  till the end while  $\tilde{\theta}$  will be eliminated, the global supersymmetry transformations are taken as  $\delta\theta = \epsilon, \delta\tilde{\theta} = 0, \delta x^m = i\epsilon\gamma^m\theta$ . Because  $\theta$  is introduced in the simple difference  $\tilde{\theta} - \theta$ , there arises an apparently trivial local fermionic invariance

under  $\delta\theta = \chi, \delta\tilde{\theta} = \chi$ , where  $\chi$  is a local fermionic parameter. If we gauge-fix  $\theta$  to be zero using this symmetry, we get back the original Brink-Schwarz action. It is easily checked that the  $\kappa$ -symmetry for  $\tilde{\theta}$  (with  $\delta\theta \equiv 0$ ) remains intact.

The standard Dirac analysis generates the following constraints:

$$\tilde{D}_\alpha = \tilde{p}_\alpha - i(\not{p}\tilde{\theta})_\alpha = 0, \quad D_\alpha = p_\alpha - i(\not{p}(\theta - 2\tilde{\theta}))_\alpha = 0, \quad (2.9)$$

$$T = \frac{1}{2}p^2 = 0. \quad (2.10)$$

$D_\alpha$  is a new constraint associated with  $\theta^\alpha$ . It is more convenient to replace it with the linear combination  $\Delta_\alpha \equiv D_\alpha + \tilde{D}_\alpha$ , which can be identified as the generator of the extra local fermionic symmetry. Since  $\Delta_\alpha$  can be easily checked to Poisson anti-commute with  $\tilde{D}_\alpha$  and with itself, the only non-vanishing bracket is

$$\{\tilde{D}_\alpha, \tilde{D}_\beta\}_P = 2i\not{p}_{\alpha\beta}. \quad (2.11)$$

The situation being exactly the same as in the BS formalism, we must employ the light-cone decomposition to identify  $\tilde{D}_a$  as the second class and the  $\kappa$ -generator  $\tilde{K}_a = \tilde{D}_a - (p^i/p^+)\gamma_{ab}^i \tilde{D}_b$  as the first class part of  $\tilde{D}_\alpha$ . They (anti-)commute with each other and with  $T$  and satisfy the relations

$$\{\tilde{D}_a, \tilde{D}_b\}_P = 2ip^+ \delta_{ab}, \quad \{\tilde{K}_a, \tilde{K}_b\}_P = -4i \frac{T}{p^+} \delta_{ab}. \quad (2.12)$$

Now by imposing the SLC gauge  $\tilde{\theta}_a = 0$ ,  $\tilde{K}_a$ 's are turned into second class and, together with  $\tilde{D}_a$ , are handled by the use of the appropriate Dirac bracket  $\{, \}_D$ . Upon this step, the remaining part of  $\tilde{\theta}$ , namely  $\tilde{\theta}_a$ , becomes self-conjugate: With a slight rescaling, we have  $S_a \equiv \sqrt{2p^+} \tilde{\theta}_a$  satisfying  $\{S_a, S_b\}_D = i\delta_{ab}$ .

The crucial difference from the BS formalism is that, under the Dirac bracket, *we still have a non-trivial first class algebra formed by  $T$  and  $D_\alpha$*  (which is the same as  $\Delta_\alpha$  since  $\tilde{D}_\alpha$  has been set strongly to zero). It reads

$$\{D_a, D_b\}_D = -4i \frac{T}{p^+} \delta_{ab}, \quad \text{rest} = 0. \quad (2.13)$$

This is identical in form to the one satisfied by  $\tilde{K}_a$  above (under the Poisson bracket) and shows that, through the new local fermionic symmetry, the content of the  $\kappa$ -symmetry for  $\tilde{\theta}$  is transferred to the sector involving the new spinor  $\theta$ .

The quantization is performed by replacing the Dirac bracket by the quantum bracket. With a slight rescaling of fields, we can set  $[x^m, p_n] = i\delta_n^m, \{p_\alpha, \theta^\beta\} = \delta_\alpha^\beta, \{S_a, S_b\} = \delta_{ab}$ . The classical algebra (2.13) then turns into the quantum algebra

$$\{D_a, D_b\} = -4i \frac{T}{p^+} \delta_{ab}, \quad \text{rest} = 0, \quad (2.14)$$

with

$$D_a = d_a + i\sqrt{2p^+} S_a, \quad D_a = d_a + i\sqrt{\frac{2}{p^+}} p^i \gamma_{ab}^i S_b, \quad (2.15)$$

where we have separated for convenience the spinor covariant derivative  $d_\alpha \equiv p_\alpha + (\not{p}\theta)_\alpha$  for the  $\theta$  sector.



It is now straightforward to construct the nilpotent BRST operator associated with the above first class algebra. It reads

$$\hat{Q} = \tilde{\lambda}^\alpha D_\alpha + \frac{2}{p^+} \tilde{\lambda}_{\dot{a}} \tilde{\lambda}_{\dot{a}} b + cT, \tag{2.16}$$

where  $\tilde{\lambda}^\alpha = (\tilde{\lambda}_a, \tilde{\lambda}_{\dot{a}})$  is an *unconstrained* bosonic spinor ghost and  $(b, c)$  are the usual fermionic ghosts satisfying  $\{b, c\} = 1$ . Note that the familiar important relation  $\{\hat{Q}, b\} = T$  holds.

The remaining task is to show that this  $\hat{Q}$  has the same cohomology as the Berkovits'  $Q = \lambda^\alpha d_\alpha$ , with the constraint  $\lambda\gamma^m\lambda = 0$ . This can be done by suitable quantum similarity transformations, which preserve the nilpotency and the cohomology. First, to remove  $T, c$  and  $b$ , we introduce an auxiliary field  $l_{\dot{a}}$  with the properties  $\tilde{\lambda}_{\dot{a}} l_{\dot{a}} = 1, l_{\dot{a}} l_{\dot{a}} = 0$  and form the following operator  $b_B$

$$b_B \equiv -\frac{p^+}{4} l_{\dot{a}} D_{\dot{a}}. \tag{2.17}$$

This may be called a composite  $b$ -ghost<sup>3</sup> in the sense that it satisfies  $\{\hat{Q}, b_B\} = T$  and  $\{b_B, b_B\} = 0$  just like  $b$ . Then by a similarity transformation  $e^{X(\star)}e^{-X}$  with  $X = b_B c$ , one can remove  $T$  and  $c$  and obtains

$$e^{X\hat{Q}}e^{-X} = \tilde{Q} + \frac{2}{p^+} \tilde{\lambda}_{\dot{a}} \tilde{\lambda}_{\dot{a}} b, \tag{2.18}$$

$$\tilde{Q} = \tilde{\lambda}_a D_a + \lambda_{\dot{a}} D_{\dot{a}}, \tag{2.19}$$

where  $\lambda_{\dot{a}} \equiv \tilde{\lambda}_{\dot{a}} - (1/2)l_{\dot{a}}\tilde{\lambda}_i\tilde{\lambda}_i$  satisfies the relation  $\lambda_{\dot{a}}\lambda_{\dot{a}} = 0$ , recognized as a part of the PS condition. Further, since  $c$  is no longer present, the last term containing  $b$  can be dropped without changing the cohomology. Note that even though  $T, b, c$  have disappeared we still have the relation  $\{\tilde{Q}, b_B\} = T$ . Finally, one can show that the non-covariant fermionic fields  $S_a$  in  $\tilde{Q}$  cohomologically decouple together with 4 of the 8 components of  $\tilde{\lambda}_a$  in such a way that the remaining 11 components of  $\tilde{\lambda}^\alpha$  precisely form a pure spinor  $\lambda^\alpha$  satisfying the condition  $\lambda\gamma^m\lambda = 0$ . This can again be effected by a similarity transformation [22] but we will not reproduce the detail here.

## 2.2 Superstring

The basic idea described above turned out to work for the superstring case as well. However, there were several new complications, which we list below and briefly describe how they were overcome.

- First, the basic Green-Schwarz action, with a modification of the form (2.8), is more non-linear due to the presence of the Wess-Zumino term. This and the existence of

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<sup>3</sup>We observe that this is very similar to the composite  $b$  ghost constructed in the recent “non-minimal” (topological) formulation of the PS formalism [10] and is expected to play an important role in deriving that theory from the first principle.

the added spinor make the separation between the left-moving and the right-moving sectors cumbersome. These complications, however, are only technical and do not cause essential problems.

- Another difference is that the various quantities now contain  $\sigma$ -derivatives. This leads to a more serious problem: The Dirac brackets among the original basic fields are no longer canonical. Fortunately, we were able to overcome this difficulty by constructing a modified basis in which the (redefined) fields satisfy canonical bracket relations.
- The third new feature in the string case is that in order to realize the crucial first class constraint algebra quantum mechanically one must make modifications due to multiple contractions and the normal-ordering of composite operators. Fortunately again, the needed modifications were minor and could be found systematically.

In the case of the supermembrane, similar complications are expected to arise. We shall see that some of them can be handled in parallel with the string case but some others present qualitatively new problems.

### 3. Double spinor formalism for supermembrane at the classical level

Having clarified the basic idea of the double spinor formalism, let us now apply it to the supermembrane case.

#### 3.1 Fundamental action and its symmetries

Just as in the superparticle and the superstring cases, the fundamental action for the double spinor formalism for the supermembrane is obtained from the conventional BST action [31] by simple replacements of fields. Setting the membrane tension to unity, it reads

$$S = \int d^3\xi (\mathcal{L}_K + \mathcal{L}_{WZ}), \tag{3.1}$$

$$\mathcal{L}_K = -\frac{1}{2}\sqrt{-g}(g^{IJ}\Pi_I^M\Pi_{JM} - 1), \tag{3.2}$$

$$\mathcal{L}_{WZ} = -\frac{1}{2}\epsilon^{IJK}W_{JMN}(\Pi_J^M\Pi_{KM} + \Pi_J^M W_{KM} + \frac{1}{3}W_J^M W_{KM}), \tag{3.3}$$

where

$$\Theta \equiv \tilde{\theta} - \theta, \quad y^M \equiv x^M - i\tilde{\theta}\Gamma^M\tilde{\theta}, \tag{3.4}$$

and the basic building blocks are defined as

$$\Pi_I^M \equiv \partial y^M - W_I^M, \tag{3.5}$$

$$W_I^M \equiv i\tilde{\Theta}\Gamma^M\partial_I\Theta, \quad W_I^{MN} \equiv i\tilde{\Theta}\Gamma^{MN}\partial_I\Theta. \tag{3.6}$$

Our notations and conventions are as follows:<sup>4</sup>  $\xi^I = (t, \sigma^i)$  ( $i = 1, 2$ ) stands for the worldvolume coordinate,  $x^M$  ( $M = 0, \dots, 10$ ) is the membrane coordinate, and  $\tilde{\theta}_A$  and

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<sup>4</sup>For more details, see appendix A.

$\theta_A$  ( $A = 1, \dots, 32$ ) are the two species of Majorana spinors. As before  $\theta$  is the newly added spinor characteristic of this formalism. The worldvolume metric is denoted by  $g_{IJ}$  ( $I = 0, 1, 2$ ). As for the  $\Gamma$ -matrices, we employ the 32-dimensional Majorana representation and denote them by  $\Gamma_{AB}^M$ . The charge conjugation matrix  $C$  equals  $\Gamma^0$  and is antisymmetric ( $C^T = -C$ ). Other combinations that frequently appear are  $C\Gamma^M$  and  $C\Gamma^{MN} \equiv C(\Gamma^M\Gamma^N - \Gamma^N\Gamma^M)/2$ , which are both symmetric. The Dirac conjugation of a spinor is defined by  $\bar{\theta}_A \equiv (\theta C)_A$ .

The symmetries possessed by the action above are essentially of the same kind as in the superparticle case. In particular, the following three fermionic symmetries will be important in the subsequent analyses:

1. Global supersymmetry:

$$\delta\tilde{\theta}_A = 0, \quad \delta\theta_A = \epsilon_A, \quad \delta x^M = i\bar{\epsilon}\Gamma^M\theta. \quad (3.7)$$

Note that the transformation closes within the  $(x^M, \theta_A)$  sector. This will allow us to gauge-fix  $\tilde{\theta}$  *without* breaking this symmetry.

2.  $\kappa$ -symmetry:

$$\begin{aligned} \delta\Theta_A &= (1 + \Gamma)\kappa(\xi), \quad \delta y^M = i\bar{\Theta}\Gamma^M(1 + \Gamma)\kappa(\xi), \\ \delta(\sqrt{-g}g^{IJ}) &= 2i(\bar{\kappa}(1 + \Gamma)\Gamma_{MN}\partial^K\Theta)g^{KI}\epsilon^{JL_1L_2}\Pi_{L_1}^M\Pi_{L_2}^N \\ &\quad + \frac{2i}{3\sqrt{-g}}(\bar{\kappa}\Gamma^M\partial^K\Theta)\Pi_{K,M}\epsilon^{LL_1L_2}\epsilon^{JL_3L_4} \\ &\quad \times (\Pi_{L_1} \cdot \Pi_{L_3}\Pi_{L_2} \cdot \Pi_{L_4} + \Pi_{L_1} \cdot \Pi_{L_3}g_{L_2L_4} + g_{L_1L_3}g_{L_2L_4}) \\ &\quad + (I \leftrightarrow J), \end{aligned} \quad (3.8)$$

where

$$\Gamma \equiv (1/3!\sqrt{-g})\epsilon^{IJK}\Pi_I^M\Pi_J^N\Pi_K^P\Gamma_{MNP}, \quad \Gamma^2 = 1 \quad (\text{on-shell}). \quad (3.9)$$

This is nothing but the standard  $\kappa$ -symmetry [31, 33] written in terms of  $(y^M, \Theta_A)$ .

3. New local fermionic symmetry:

$$\delta\tilde{\theta}_A = \delta\theta_A = \chi_A, \quad \delta x^M = i(\bar{\chi}\Gamma^M\Theta), \quad (\delta\Theta_A = \delta y^M = 0), \quad (3.10)$$

where  $\chi_A(\xi)$  is a local fermionic parameter. Clearly, one can use this symmetry to gauge-fix  $\theta_A$  to zero, upon which our action reduces to the conventional BST action. On the other hand, if we keep this local symmetry till the end it is expected to lead to a first class algebra of constraints, which was the pivotal element for deriving the PS formalism for the superparticle and the superstring [22]. Below we shall investigate if this will be the case for the supermembrane as well.

### 3.2 Analysis of constraints

As we shall use the Hamiltonian formulation, it is most efficient to employ the ADM decomposition of the worldvolume metric [34], namely,  $ds^2 = -(Ndt)^2 + \gamma_{ij}(d\sigma^i + N^i dt)(d\sigma^j + N^j dt)$ , where  $N, N^i$  and  $\gamma_{ij}$  are, respectively, the lapse, the shift and the spatial metric. We will also use the notation  $g \equiv \det g_{ij} = N\sqrt{\gamma}$ , where  $\gamma \equiv \det \gamma_{ij}$ . In terms of these ADM variables our action can be written as

$$\mathcal{L}_K = \frac{1}{2}A\Pi_0^M\Pi_{0M} + B^M\Pi_{0M} + C, \quad (3.11)$$

$$\mathcal{L}_{WZ} = F_M\Pi_0^M + \tilde{\Phi}_A\dot{\Theta}_A, \quad (3.12)$$

where<sup>5</sup> the quantities

$$A = \frac{\sqrt{\gamma}}{N}, \quad B^M = -AN^i\Pi_i^M, \quad C = \frac{A}{2}N^2(1 - \gamma^{ij}\Pi_i \cdot \Pi_j) + \frac{B^2}{2A}, \quad (3.13)$$

$$F_M = \epsilon^{ij}W_{iMN}\left(\Pi_j^N + \frac{1}{2}W_j^N\right), \quad (\epsilon^{12} = 1), \quad (3.14)$$

are bosonic and

$$\begin{aligned} \tilde{\Phi}_A = & -\frac{i}{2}\epsilon^{ij}(\bar{\Theta}\Gamma_{MN})_A\left(\Pi_i^M\Pi_j^N + \Pi_i^M W_j^N + \frac{1}{3}W_i^M W_j^N\right) \\ & + \frac{i}{2}\epsilon^{ij}W_{iMN}(\bar{\Theta}\Gamma^M)_A\left(\Pi_j^N + \frac{2}{3}W_j^N\right) \end{aligned} \quad (3.15)$$

is fermionic. It is important to note that  $\Pi_i^M, F_M, \tilde{\Phi}_A, W_i^M$  and  $W_{iMN}$  are invariant under the local fermionic symmetry (3.10).

We will denote the canonical conjugates to the basic variables  $(N, N^i, \gamma_{ij}, x^M, \tilde{\theta}_A, \theta_A)$  by  $(P, P_i, P^{ij}, k_M, \tilde{k}_A, k_A)$ . They are defined in the standard manner such as  $k_A \equiv (\partial/\partial\theta_A)\mathcal{L}$ , where for fermions we use left derivatives. The Poisson brackets for the fundamental fields are taken as

$$\{N(\sigma), P(\sigma')\}_P = \delta(\sigma - \sigma'), \quad \{N^i(\sigma), P_j(\sigma')\}_P = \delta_j^i\delta(\sigma - \sigma'), \quad (3.16)$$

$$\{\gamma_{ij}(\sigma), P^{k\ell}(\sigma')\}_P = \delta_{ij}^{k\ell}\delta(\sigma - \sigma'), \quad (\delta_{ij}^{k\ell} \equiv \delta_i^k\delta_j^\ell - \delta_i^\ell\delta_j^k), \quad (3.17)$$

$$\{x^M(\sigma), k^N(\sigma')\}_P = \eta^{MN}\delta(\sigma - \sigma'), \quad (3.18)$$

$$\{\tilde{\theta}_A(\sigma), \tilde{k}_B(\sigma')\}_P = \{\theta_A(\sigma), k_B(\sigma')\}_P = -\delta_{AB}\delta(\sigma - \sigma'), \quad (3.19)$$

$$\text{rest} = 0. \quad (3.20)$$

Since  $A, B, C, F$  and  $\tilde{\Phi}_A$  do not contain time derivatives, these canonical conjugates are readily computed and some of them lead to the primary constraints. First, as the action does not contain the time derivative of the worldvolume metric, their conjugates  $(P, P_i, P^{ij})$  vanish. Similarly, the definitions of the fermionic momenta  $\tilde{k}_A$  and  $k_A$  lead to the constraints

$$\tilde{D}_A \equiv \tilde{k}_A - ik_M(\bar{\theta}\Gamma^M)_A + \tilde{\Phi}_A \approx 0, \quad (3.21)$$

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<sup>5</sup>We define  $X^2 \equiv X^M X_M$  and  $X \cdot Y \equiv X^M Y_M$ .

$$D_A \equiv k_A + ik_M((2\bar{\theta} - \bar{\theta})\Gamma^M)_A - \tilde{\Phi}_A \approx 0, \quad (3.22)$$

where  $\approx 0$  means weakly zero. The highly complicated quantity  $\tilde{\Phi}_A$  disappears in the sum

$$\Delta_A \equiv \tilde{D}_A + D_A = \tilde{k}_A + k_A + ik_M(\bar{\Theta}\Gamma^M)_A, \quad (3.23)$$

which can be identified as the generator of the local fermionic symmetry. Below, we shall take  $\tilde{D}_A$  and  $\Delta_A$  as the independent set of constraints, instead of  $\tilde{D}_A$  and  $D_A$ . After some algebra, the Poisson brackets among  $\tilde{D}_A$  and  $\Delta_A$  are found as

$$\{\tilde{D}_A(\sigma), \tilde{D}_B(\sigma')\}_P = iG_{AB}\delta(\sigma - \sigma'), \quad (3.24)$$

$$\{\Delta_A(\sigma), \Delta_B(\sigma')\}_P = \{\Delta_A(\sigma), \tilde{D}_B(\sigma')\}_P = 0, \quad (3.25)$$

where  $G_{AB}$  is given by

$$G_{AB} \equiv 2\mathcal{K}_M(C\Gamma^M)_{AB} + \epsilon^{ij}\Pi_i^M\Pi_j^N(C\Gamma_{MN})_{AB}. \quad (3.26)$$

The first term on the RHS is the counterpart of the operator  $\not{p}_{\alpha\beta}$  in the superparticle case, while the second term is a new structure characteristic of the supermembrane.

Now, as is usual, we must see if the constraints are consistent with the time development. The canonical Hamiltonian is given, up to the constraints described above, by

$$H = \int d^2\sigma \mathcal{H}, \quad \mathcal{H} = \frac{N}{\sqrt{\gamma}}\mathcal{T} + N^i\mathcal{T}_i, \quad (3.27)$$

where

$$\begin{aligned} \mathcal{T} &\equiv \frac{1}{2}(\mathcal{K}^2 + \gamma(\gamma^{ij}\Pi_i \cdot \Pi_j - 1)), & \mathcal{T}_i &\equiv \mathcal{K} \cdot \Pi_i, \\ \mathcal{K}^M &\equiv k^M - F^M. \end{aligned} \quad (3.28)$$

$\mathcal{T}$  and  $\mathcal{T}_i$  are the generators of the worldvolume reparametrization. Demanding the consistency of the vanishing of  $(P, P_i, P^{ij})$  with the time development, we get the secondary constraints

$$T^{(0)} \equiv \mathcal{K}^2 + M (\approx 2\mathcal{T}) \approx 0, \quad T_i^{(1)} \equiv \mathcal{T}_i \approx 0, \quad T_{ij}^{(2)} \equiv \Pi_i \cdot \Pi_j - \gamma_{ij} \approx 0, \quad (3.29)$$

where  $M_{ij} \equiv \Pi_i \cdot \Pi_j$ ,  $M \equiv \det M_{ij}$ . Therefore, the total Hamiltonian at this stage consists purely of constraints

$$\begin{aligned} H_T &= \int d^2\sigma \mathcal{H}_T, \\ \mathcal{H}_T &= uP + u_i P_i + u_{ij} P^{ij} \\ &\quad + u^{(0)}T^{(0)} + u_i^{(1)}T_i^{(1)} + u_{ij}^{(2)}T_{ij}^{(2)} + \tilde{v}^\alpha \tilde{D}_\alpha + v^\alpha \Delta_\alpha, \end{aligned} \quad (3.30)$$

where the Lagrange multipliers  $u$ 's and  $v$ 's are arbitrary functions on the phase space. We must check if the constraints are maintained in time by computing their Poisson brackets with  $H_T$ . It turned out that our system is already consistent and no new constraints arise: With appropriate reorganization, all the constraints weakly commute with  $H_T$ . This will become evident in the next subsection, where we consider the lightcone decomposition of the constraints.

### 3.2.1 Separation of first and second class constraints

Now we must perform the separation of the first and the second class constraints. This is done by (block) diagonalizing the matrix  $C_{IJ}$  given by

$$C_{IJ} \equiv \{\phi_I, \phi_J\}_P, \quad \phi_I = (P, P_i, P^{ij}, T^{(0)}, T_i^{(1)}, T_{ij}^{(2)}, \tilde{D}_\alpha, \Delta_\alpha), \quad (3.31)$$

on the constraint surface  $\phi_I = 0$ . After some analysis, we find that the following set of constraints, equivalent to the original ones, do the job:

$$P, \quad P_i, \quad P^{ij}, \quad T_{ij}^{(2)}, \quad \Delta_\alpha, \quad (3.32)$$

$$\hat{D}_A \equiv \tilde{D}_A - 4P^{ij}(C\Gamma^M \partial_i \tilde{\theta})_A \Pi_{jM}, \quad (3.33)$$

$$\hat{T}^{(0)} \equiv T^{(0)} - 2P^{ij} \partial_i \Pi_j \cdot \mathcal{K} + i\epsilon^{ij} (\partial_i \tilde{\theta} \Gamma^M \tilde{D}) \Pi_{jM}, \quad (3.34)$$

$$\hat{T}_i^{(1)} \equiv T_i^{(1)} - 2\gamma_{jk} \partial_i P^{jk} + 2\partial_i \Theta_A \tilde{D}_A. \quad (3.35)$$

The brackets among them all vanish on the constraint surface, except the following two:

$$\{P^{ij}(\sigma), T_{kl}^{(2)}(\sigma')\}_P = \delta_{kl}^{ij} \delta(\sigma - \sigma'), \quad (3.36)$$

$$\{\hat{D}_A(\sigma), \hat{D}_B(\sigma')\}_P \approx iG_{AB} \delta(\sigma - \sigma'). \quad (3.37)$$

$G_{AB}$ , as defined in (3.26), has rank 16. Therefore,  $P^{ij}$ ,  $T_{ij}^{(2)}$  and a half of  $\hat{D}_A$  are of second class and all the others are of first class. Actually, the pair of second class constraints  $(P^{ij}, T_{ij}^{(2)})$  commute with others even in the strong sense. This means that the Dirac bracket on the constraint surface  $P^{ij} = T_{ij}^{(2)} = 0$  is identical to the original Poisson bracket, and we may ignore them together with their conjugates  $(\gamma_{ij}, P^{ij})$ . By choosing the gauge in which  $N = 1$  and  $N^i = 0$ ,  $P$  and  $P_i$  can be disposed of by the same reason.

Thus we are left with the remaining constraints

$$\tilde{D}_A = \tilde{k}_A - ik_M (\tilde{\theta} \Gamma^M)_A + \tilde{\Phi}_A, \quad (3.38)$$

$$T^{(0)} = \mathcal{K}^2 + M + i\epsilon^{ij} (\partial_i \tilde{\theta} \Gamma^M \tilde{D}) \Pi_{jM}, \quad (3.39)$$

$$T_i^{(1)} = \mathcal{K} \cdot \Pi_i + 2\partial_i \Theta_A \tilde{D}_A, \quad (3.40)$$

$$\Delta_\alpha = \tilde{k}_A + k_A + ik_M (\tilde{\Theta} \Gamma^M)_A, \quad (3.41)$$

where for simplicity we removed the hats from the modified constraints  $(\hat{D}_A, \hat{T}^{(0)}, \hat{T}^{(1)})$ . Except for a half of  $\tilde{D}_A$ , all of them are first class:  $T^{(0)}$  and  $T_i^{(1)}$  generate the worldvolume reparametrizations,  $\Delta_\alpha$  generates the local fermionic symmetry, and the first class part of  $\tilde{D}_A$  generates the  $\kappa$ -symmetry (3.24).

We shall now separate the first and the second class part of  $\tilde{D}_A$  explicitly by defining the generator of  $\kappa$ -symmetry. This can be done most efficiently by making the following lightcone decomposition. We take a basis of spinors in which the lightcone chirality operator (i.e. the  $SO(1, 1)$  boost charge) given by  $\hat{\Gamma} = \Gamma^0 \Gamma^{10}$  is diagonal and decompose spinors into lightcone chiral and anti-chiral components according to their  $\hat{\Gamma}$  eigenvalues:  $\phi_A = (\phi_\alpha, \phi_{\dot{\alpha}})$ ,  $\hat{\Gamma}_{\alpha\beta} \phi_\beta = \phi_\alpha$  and  $\hat{\Gamma}_{\dot{\alpha}\dot{\beta}} \phi_{\dot{\beta}} = -\phi_{\dot{\alpha}}$ . It will be useful to remember that  $C\Gamma^\pm \equiv C(\Gamma^0 \pm \Gamma^{10})$ , with

non-vanishing components  $C\Gamma_{\dot{\alpha}\dot{\beta}}^+ = -2\delta_{\dot{\alpha}\dot{\beta}}$ ,  $C\Gamma_{\alpha\beta}^- = -2\delta_{\alpha\beta}$ , serve essentially as projectors. As for vectors, the decomposition is defined as  $v^M = (v^+, v^-, v^m)$ ,  $v^\pm \equiv v^0 \pm v^{10}$ ,  $m = 1, \dots, 9$ . (Further details of our conventions can be found in appendix A.)

In this basis,  $G_{AB}$  decomposes as

$$G_{\alpha\beta} = 2(A + B_m\gamma^m)_{\alpha\beta}, \quad G_{\dot{\alpha}\dot{\beta}} = 2(A' + B'_m\gamma^m)_{\dot{\alpha}\dot{\beta}}, \quad (3.42)$$

$$G_{\alpha\dot{\beta}} = (C + D_m\gamma^m + E_{mn}\gamma^{mn})_{\alpha\dot{\beta}}, \quad G_{\dot{\alpha}\beta} = (C + D_m\gamma^m - E_{mn}\gamma^{mn})_{\dot{\alpha}\beta}, \quad (3.43)$$

where

$$A = \mathcal{K}^+, \quad B^m = -\epsilon^{ij}\Pi_i^+\Pi_j^m, \quad (3.44)$$

$$A' = \mathcal{K}^-, \quad B'^m = \epsilon^{ij}\Pi_i^-\Pi_j^m, \quad (3.45)$$

$$C = \epsilon^{ij}\Pi_i^-\Pi_j^+, \quad D_m = 2\mathcal{K}_m, \quad E_{mn} = \epsilon^{ij}\Pi_{im}\Pi_{jn}. \quad (3.46)$$

Here and hereafter, we assume that  $A = \mathcal{K}^+$ ,  $B \equiv \sqrt{B^m B^m}$  and  $A^2 - B^2$  are non-vanishing. This is the analogue of the usual assumption in the Green-Schwarz superstring that the lightcone momentum  $k^+$  does not vanish. Now, to construct the  $\kappa$ -generator, we will need the inverse of the matrix  $G_{\alpha\beta}$ . Consider the quantity  $\hat{u}_{\alpha\beta} \equiv \frac{1}{B}B^m\gamma_{\alpha\beta}^m$ . It is a symmetric matrix satisfying  $\hat{u}^2 = B_m B_n \gamma^m \gamma^n / B^2 = 1$ , due to the Clifford algebra. Therefore, the set of expressions of the type  $(x + y\hat{u})_{\alpha\beta}$  appearing in  $G_{\alpha\beta}$  form an algebraic field. This immediately allows us to compute the the inverse as<sup>6</sup>

$$G_{\alpha\beta}^{-1} = \frac{1}{(A^2 - B^2)}(A - B\hat{u})_{\alpha\beta}. \quad (3.47)$$

Now the  $\kappa$ -generator  $\tilde{K}_{\dot{\alpha}}$ , i.e. the first class part of  $\tilde{D}_A$ , can be identified as the following linear combination of the constraints  $\tilde{D}_\alpha$  and  $\tilde{D}_{\dot{\alpha}}$

$$\tilde{K}_{\dot{\alpha}} \equiv \tilde{D}_{\dot{\alpha}} - G_{\dot{\alpha}\delta}(G^{-1})_{\delta\gamma}\tilde{D}_\gamma. \quad (3.48)$$

After lengthy but straightforward computations, its Poisson brackets with  $\tilde{D}_\alpha$  and with itself are found as

$$\{\tilde{K}_{\dot{\alpha}}(\sigma), \tilde{D}_\beta(\sigma')\}_P = (\tilde{D}\text{-terms}) \approx 0, \quad (3.49)$$

$$\{\tilde{K}_{\dot{\alpha}}(\sigma), \tilde{K}_\beta(\sigma')\}_P = \mathcal{T}_{\dot{\alpha}\dot{\beta}}\delta(\sigma - \sigma') + (\tilde{D}\text{-terms}) \approx 0, \quad (3.50)$$

where

$$\mathcal{T}_{\dot{\alpha}\dot{\beta}} \equiv \mathcal{T}\delta_{\dot{\alpha}\dot{\beta}} + \mathcal{T}_m\gamma_{\dot{\alpha}\dot{\beta}}^m, \quad (3.51)$$

$$\mathcal{T} \equiv \frac{2}{i(A^2 - B^2)}(AT^{(0)} - 2B_m\epsilon^{ij}\Pi_i^m T_j^{(1)} + C\epsilon^{ij}\Pi_i^+ T_j^{(1)}), \quad (3.52)$$

$$\mathcal{T}_m \equiv \frac{2}{i(A^2 - B^2)}(B_m T^{(0)} - 2A\epsilon^{ij}\Pi_{im} T_j^{(1)} + 2\mathcal{K}_m\epsilon^{ij}\Pi_i^+ T_j^{(1)}). \quad (3.53)$$

---

<sup>6</sup>Note that  $G_{\alpha\beta}^{-1}$  differs from the  $\alpha\beta$ -component of the inverse of the full matrix  $G_{AB}$ .

Here “ $\tilde{D}$ -terms” are those which vanish upon imposing the fermionic constraint  $\tilde{D}_A = 0$ . Since both  $\mathcal{T}$  and  $\mathcal{T}_m$  are linear combinations of the original bosonic constraints  $T^{(0)}$  and  $T_i^{(1)}$ ,  $\tilde{K}_{\dot{\alpha}}$  is indeed of first class.

This completes the classification of the constraints present in our Hamiltonian system. The net result may be summarized as follows. The constraints are classified as

$$\text{first class: } P, P_i, \Delta_A, T^{(0)}, T_i^{(1)}, \tilde{K}_{\dot{\alpha}}, \tag{3.54}$$

$$\text{second class: } P^{ij}, T_{ij}^{(2)}, \tilde{D}_{\alpha}, \tag{3.55}$$

where  $T^{(0)}, T_i^{(1)}$  are the slightly redefined expressions displayed in (3.39), (3.40) and  $\tilde{K}_{\dot{\alpha}}$  is given in (3.48). The total Hamiltonian consists entirely of first class constraints and is given by

$$H_T = \int d^2\sigma \mathcal{H}_T, \tag{3.56}$$

$$\mathcal{H}_T = uP + u^i P_i + u_A \Delta_A + u^{(0)} T^{(0)} + u_i^{(1)} T_i^{(1)} + \tilde{u}^{\dot{\alpha}} \tilde{K}_{\dot{\alpha}}. \tag{3.57}$$

### 3.3 First class algebra in the SLC gauge

#### 3.3.1 Dirac bracket in the SLC gauge

Up to this point, we have not gauge-fixed any of the local symmetries of the system. Now we fix the  $\kappa$ -symmetry generated by  $\tilde{K}_{\dot{\alpha}}$  by imposing the SLC gauge

$$\Gamma^+ \tilde{\theta} = 0 \quad \Leftrightarrow \quad \tilde{\theta}_{\dot{\alpha}} = 0. \tag{3.58}$$

This renders the pair  $(\tilde{\theta}_{\dot{\alpha}}, \tilde{K}_{\dot{\alpha}})$  second class. To define the Dirac bracket on the constraint surface<sup>7</sup> specified by

$$\phi_{\tilde{I}} \equiv (\tilde{D}_{\alpha}, \tilde{K}_{\dot{\alpha}}, \tilde{\theta}_{\dot{\alpha}}) = 0, \tag{3.59}$$

one must compute the inverse of the matrix

$$C_{\tilde{I}\tilde{J}} \equiv \{\phi_{\tilde{I}}(\sigma), \phi_{\tilde{J}}(\sigma')\}_P. \tag{3.60}$$

From (3.49), (3.50) and a simple bracket relation  $\{\tilde{K}_{\dot{\alpha}}(\sigma), \tilde{\theta}_{\dot{\beta}}(\sigma')\}_P = -\delta_{\dot{\alpha}\dot{\beta}}\delta(\sigma - \sigma')$ ,  $C_{\tilde{I}\tilde{J}}$  can be readily computed. Upon imposing  $\phi_{\tilde{I}} = 0$ , it reads

$$C_{\tilde{I}\tilde{J}} = \begin{matrix} & \tilde{D}_{\beta} & \tilde{K}_{\dot{\beta}} & \tilde{\theta}_{\dot{\beta}} \\ \begin{matrix} \tilde{D}_{\alpha} \\ \tilde{K}_{\dot{\alpha}} \\ \tilde{\theta}_{\dot{\alpha}} \end{matrix} & \begin{pmatrix} iG_{\alpha\beta} & 0 & 0 \\ 0 & \mathcal{T}_{\dot{\alpha}\dot{\beta}} & -\delta_{\dot{\alpha}\dot{\beta}} \\ 0 & -\delta_{\dot{\alpha}\dot{\beta}} & 0 \end{pmatrix} \end{matrix} \delta(\sigma - \sigma'), \tag{3.61}$$

---

<sup>7</sup>As we remarked earlier, we may simply ignore the constraints  $(P, P_i, P^{ij}, T_{ij}^{(2)})$  in the subsequent analysis.



and its inverse takes the form

$$(C^{-1})_{\bar{i}\bar{j}} = \begin{pmatrix} \tilde{D}_\beta & \tilde{K}_{\dot{\beta}} & \tilde{\theta}_{\dot{\beta}} \\ \tilde{D}_\alpha & -i(G^{-1})_{\alpha\beta} & 0 \\ \tilde{K}_{\dot{\alpha}} & 0 & 0 \\ \tilde{\theta}_{\dot{\alpha}} & 0 & -\delta_{\dot{\alpha}\dot{\beta}} \\ & 0 & -\mathcal{T}_{\dot{\alpha}\dot{\beta}} \end{pmatrix} \delta(\sigma - \sigma'). \quad (3.62)$$

Thus the Dirac bracket on the surface  $\phi_{\bar{i}} = 0$  becomes

$$\begin{aligned} & \{A(\sigma), B(\sigma')\}_D \\ & \equiv \{A(\sigma), B(\sigma')\}_P - \int d^2\sigma_1 \{A(\sigma), \phi_{\bar{i}}(\sigma_1)\}_P (C^{-1})_{\bar{i}\bar{j}}(\sigma_1) \{\phi_{\bar{j}}(\sigma_1), B(\sigma')\}_P \\ & = \{A(\sigma), B(\sigma')\}_P + i \int d^2\sigma_1 \{A(\sigma), \tilde{D}_\alpha(\sigma_1)\}_P (G^{-1})_{\alpha\beta}(\sigma_1) \{\tilde{D}_\beta(\sigma_1), B(\sigma')\}_P \\ & \quad + \int d^2\sigma_1 \{A(\sigma), \tilde{\theta}_{\dot{\alpha}}(\sigma_1)\}_P \mathcal{T}_{\dot{\alpha}\dot{\beta}}(\sigma_1) \{\tilde{\theta}_{\dot{\beta}}(\sigma_1), B(\sigma')\}_P \\ & \quad + \int d^2\sigma_1 \left( \{A(\sigma), \tilde{\theta}_{\dot{\alpha}}(\sigma_1)\}_P \{\tilde{K}_{\dot{\alpha}}(\sigma_1), B(\sigma')\}_P \right. \\ & \quad \quad \left. + \{A(\sigma), \tilde{K}_{\dot{\alpha}}(\sigma_1)\}_P \{\tilde{\theta}_{\dot{\alpha}}(\sigma_1), B(\sigma')\}_P \right). \end{aligned} \quad (3.63)$$

Now, using this bracket, we compute the algebra satisfied by the remaining first class constraints  $(\Delta_A, T^{(0)}, T_i^{(1)})$ , or equivalently, by  $(D_A, T^{(0)}, T_i^{(1)})$ . As in the case of the superparticle and the superstring, we expect that the information of the  $\kappa$ -symmetry will be reflected in the brackets involving  $D_{\dot{\alpha}}$ . More explicitly, the Dirac bracket  $\{D_{\dot{\alpha}}, D_{\dot{\beta}}\}_D$  would produce  $\mathcal{T}_{\dot{\alpha}\dot{\beta}}$  just as in the Poisson bracket  $\{\tilde{K}_{\dot{\alpha}}, \tilde{K}_{\dot{\beta}}\}_P$  given in (3.50). To check this, first note that under the Dirac bracket  $\{D_A, D_B\}_D$  is equal to  $\{\Delta_A, \Delta_B\}_D$ . Further, since both  $\tilde{D}_\alpha$  and  $\tilde{K}_{\dot{\alpha}}$  as well as  $\Delta_A$  itself are invariant under the local fermionic symmetry, we have

$$\{\Delta_A, \tilde{D}_\alpha\}_P = \{\Delta_A, \tilde{K}_{\dot{\alpha}}\}_P = \{\Delta_A, \Delta_B\}_P = 0. \quad (3.64)$$

Using these relations we easily find

$$\{D_{\dot{\alpha}}(\sigma), D_{\dot{\beta}}(\sigma')\}_D = \{\Delta_{\dot{\alpha}}(\sigma), \Delta_{\dot{\beta}}(\sigma')\}_D = (\mathcal{T}\delta_{\dot{\alpha}\dot{\beta}} + \mathcal{T}_m \gamma_{\dot{\alpha}\dot{\beta}}^m) \delta(\sigma - \sigma'), \quad (3.65)$$

$$\{D_A(\sigma), D_B(\sigma')\}_D = \{\Delta_A(\sigma), \Delta_B(\sigma')\}_D = 0, \quad (3.66)$$

which are of the expected form. The relation (3.65) is quite analogous to the corresponding formula  $\{\tilde{D}_{\dot{a}}, \tilde{D}_{\dot{b}}\}_D = (-4i/p^+) T \delta_{\dot{a}\dot{b}}$  for the superparticle, but there is one crucial difference: While the constraint  $T = p^2/2$  in the superparticle case simply commutes with itself under the Dirac bracket, this is not the case for the reparametrization generators  $T^{(0)}$  and  $T_i^{(1)}$  for the supermembrane. As this situation is very similar to the one we encountered in the analysis of the PS superstring, perhaps it is useful to recall briefly how this issue was resolved in that case.

### 3.3.2 A brief revisit to the superstring case

At the corresponding stage in the analysis of the type II superstring in the double spinor formalism, we were left with the first class constraints  $(\Delta_\alpha, \hat{\Delta}_\alpha, T^{(0)}, T^{(1)})$ , where  $\Delta_\alpha$  and  $\hat{\Delta}_\alpha$  are the generators of the local fermionic symmetry for the left/right sectors, and  $T^{(0)}$  and  $T^{(1)}$  are the worldsheet reparametrization generators. The combinations  $T \equiv T^{(0)} + T^{(1)}$ ,  $\hat{T} \equiv T^{(1)} - T^{(0)}$  generate left/right Virasoro algebras and one obtains a nice orthogonal split:

$$\begin{aligned} L \text{ (left): } & (\Delta_\alpha, T), & R \text{ (right): } & (\hat{\Delta}_\alpha, \hat{T}), \\ \Rightarrow & \{L, R\}_D = 0. \end{aligned} \tag{3.67}$$

Below, we will exclusively deal with the left sector.

In this sector, we obtained the Dirac bracket relation, analogous to (3.65) above, of the form

$$\{D_{\dot{a}}(\sigma), D_{\dot{b}}(\sigma')\}_D = 4iT\delta_{\dot{a}\dot{b}}\delta(\sigma - \sigma'), \quad \mathcal{T} \equiv T/\Pi^+, \tag{3.68}$$

where  $\dot{a}, \dot{b}$  denote the  $SO(8)$  anti-chiral indices and  $\Pi^+$  is a component of the superinvariant momentum  $\Pi^m = k^m + \partial_\sigma x^m + \dots$ , which is assumed non-vanishing as usual. A remarkable fact was that the operator  $\mathcal{T}$  on the RHS turned out to have a much nicer property than  $T$ . While  $T$  generates the non-trivial Virasoro algebra,  $\mathcal{T}$  has vanishing Dirac brackets with  $D_\alpha$  and with itself, precisely because of the presence of the denominator  $\Pi^+$ . Moreover, since  $\Pi^+ \neq 0$ , obviously  $\mathcal{T}$  and  $T$  impose the same phase space constraint. Therefore, even when the algebra of reparametrization is non-trivial, the crucial algebra formed by the first class constraints continues to exhibit a very simple structure. We will now demonstrate that this feature persists for the supermembrane case as well.

### 3.3.3 Fundamental constraint algebra for the supermembrane

Let us now return to the supermembrane theory. From the experience with the superstring case just reviewed, we expect that, despite their apparent complexities,  $\mathcal{T}$  and  $\mathcal{T}_m$  would commute among themselves and with  $D_A$  under the Dirac bracket. However, the demonstration by direct computations requires a considerable amount of work and is unwieldy. Fortunately, there is a much more efficient way, making use of the symmetry structure of the theory.

The crucial observation is that  $\Delta_A$ , being the generator of the extra local fermionic symmetry, Poisson-commutes with all the quantities, such as  $\mathcal{K}^M$ ,  $\Pi_i^M$ ,  $F_M$ ,  $\tilde{\Phi}_A$ ,  $W_i^M$ ,  $W_{iMN}$ , etc., which are invariant under such a symmetry. This then implies that even under the Dirac bracket  $\Delta_A$  commutes with such invariants as long as they have vanishing Poisson brackets with  $\tilde{\theta}_{\dot{\alpha}}$ , i.e. as long as they are free of  $\tilde{k}_{\dot{\alpha}}$ . In particular, it is easy to check that  $\mathcal{T}$  and  $\mathcal{T}_m$  are such a quantities and hence we deduce  $\{\Delta_A, \mathcal{T}\}_D = \{\Delta_A, \mathcal{T}_m\}_D = 0$ . But since  $\Delta_A = D_A$  in the SLC gauge, this is equivalent to

$$\{D_A, \mathcal{T}\}_D = \{D_A, \mathcal{T}_m\}_D = 0. \tag{3.69}$$

As for the bracket between  $\mathcal{T}_{\dot{\alpha}\dot{\beta}}$ 's, we can make use of the representation

$$\mathcal{T}_{\dot{\alpha}\dot{\beta}}(\sigma) = \int d^2\rho \{D_{\dot{\alpha}}(\sigma), D_{\dot{\beta}}(\rho)\}_D. \quad (3.70)$$

Then,

$$\{\mathcal{T}_{\dot{\alpha}\dot{\beta}}(\sigma), \mathcal{T}_{\dot{\gamma}\dot{\delta}}(\sigma')\}_D = \int d^2\rho \{ \{D_{\dot{\alpha}}(\sigma), D_{\dot{\beta}}(\rho)\}_D, \mathcal{T}_{\dot{\gamma}\dot{\delta}}(\sigma') \}_D, \quad (3.71)$$

and this vanishes by the use of the graded Jacobi identity and (3.69). Moreover, it is clear that  $\mathcal{T}$  and  $\mathcal{T}_m$  can be independently separated from  $\mathcal{T}_{\dot{\alpha}\dot{\beta}}$ , as the unit matrix and  $\gamma^m$  are orthogonal under the trace-norm. Hence, we get

$$\{\mathcal{T}, \mathcal{T}\}_D = \{\mathcal{T}, \mathcal{T}_m\}_D = \{\mathcal{T}_m, \mathcal{T}_n\}_D = 0. \quad (3.72)$$

We have thus found a pleasing result: *The Dirac bracket algebra of the first class constraints,  $\{D_A = (D_{\dot{\alpha}}, D_{\dot{\beta}}), \mathcal{T}, \mathcal{T}_m\}$ , that governs the entire classical dynamics of the supermembrane in the double spinor formalism is of simple structure given by*

$$\{D_{\dot{\alpha}}, D_{\dot{\beta}}\}_D = \mathcal{T}\delta_{\dot{\alpha}\dot{\beta}} + \mathcal{T}_m\gamma_{\dot{\alpha}\dot{\beta}}^m, \quad (3.73)$$

$$\text{all others} = 0. \quad (3.74)$$

As in the case of the superstring, this should serve as the platform upon which to develop the pure spinor type formalism.

We must however note that there is one conspicuous difference from the string case: The term  $\mathcal{T}_m\gamma_{\dot{\alpha}\dot{\beta}}^m$  on the RHS of (3.73), which comes from the structure  $\epsilon^{ij}\Pi_i^M\Pi_j^N(C\Gamma_{MN})_{AB}$  in  $G_{AB}$  (see (3.26)), is new for the supermembrane. Consequently, the number of bosonic constraints  $\mathcal{T} = \mathcal{T}_m = 0$  appears to be more than that of the original constraints  $T^{(1)} = T_i^{(2)} = 0$ . In appendix C, we show that nevertheless the phase space defined by these two sets are equivalent under generic conditions. This in turn implies that there must be 7 linear relations among the 10 constraints  $\mathcal{T}_{\bar{m}} \equiv (\mathcal{T}_0 = \mathcal{T}, \mathcal{T}_m)$ , namely

$$Z_{\bar{p}}^{\bar{m}}\mathcal{T}_{\bar{m}} = 0, \quad \bar{p} = 1, 2, \dots, 7, \quad (3.75)$$

where  $Z_{\bar{p}}^{\bar{m}}$  are field-dependent coefficients. Thus, by definition the above algebra is of reducible type. To learn the order of the reducibility, one must find  $Z_{\bar{p}}^{\bar{m}}$  and study its properties. After some analysis we find that (3.75) splits into

$$Z_{\bar{p}}^0 = 0, \quad Z_{\bar{p}}^m\mathcal{T}_m = 0, \quad (3.76)$$

where  $Z_{\bar{p}}^m$ , given explicitly in appendix D, are linearly independent as seven 9-vectors  $(Z_{\bar{p}})^m$ . This shows that the reducibility is of first order.

Due to this reducibility of the algebra, the construction of the associated BRST charge will be more involved than for the string case. Fortunately, however, there already exists a general theory [35] to handle such a situation. We will now describe how it can be applied to our case.

### 3.4 Construction of the BRST operator

As usual, one starts by introducing the ghosts  $\eta_{a_0} = (\tilde{\lambda}_A, \tilde{c}, \tilde{c}_m)$  and their conjugate antighosts  $\varphi_{a_0} = (\tilde{\omega}_A, \tilde{b}, \tilde{b}_m)$  corresponding to the constraints  $G_{a_0} = (D_A, \mathcal{T}, \mathcal{T}_m)$ .  $(\tilde{\lambda}_A, \tilde{\omega}_A)$  are bosonic, while  $(\tilde{c}, \tilde{b})$  and  $(\tilde{c}_m, \tilde{b}_m)$  are fermionic and they are assumed to satisfy the canonical Dirac bracket relations such as  $\{\tilde{\lambda}_A(\sigma), \tilde{\omega}_B(\sigma')\}_D = \delta_{AB}\delta(\sigma - \sigma')$ ,  $\{\tilde{c}(\sigma), \tilde{b}(\sigma')\}_D = \delta(\sigma - \sigma')$ , etc. What will be the crucial book-keeping device is *the antighost number*,  $\overline{\text{gh}}\#$ , which at this stage is assigned to be 1 for  $(\tilde{\omega}_A, \tilde{b}, \tilde{b}_m)$  and 0 for all the others, including the ghosts. On the other hand, the usual ghost number  $\text{gh}\#$  is taken to be 1 for the ghosts and  $-1$  for the antighosts. The basic strategy for constructing the BRST operator  $Q$  carrying  $\text{gh}\# = 1$  is to decompose it according to the antighost number as  $Q = Q_0 + Q_1 + \dots$  and determine  $Q_n$  order by order by requiring that (i)  $Q$  is nilpotent under the Dirac bracket and (ii) its cohomology correctly realizes the gauge invariant functions defined on the constrained surface. Referring the reader to [35] for the full details and justifications, below we will explicitly describe the procedure for our system.

At  $\overline{\text{gh}}\# = 0$ , we start with

$$Q_0 = \eta_{a_0} G_{a_0} = \tilde{\lambda}_A D_A + \tilde{c} \mathcal{T} + \tilde{c}_m \mathcal{T}_m. \quad (3.77)$$

This is not nilpotent since

$$\{Q_0, Q_0\}_D = \tilde{\lambda}_{\dot{\alpha}} \tilde{\lambda}_{\dot{\alpha}} \mathcal{T} + \tilde{\lambda}_{\dot{\alpha}} \gamma_{\dot{\alpha}\dot{\beta}}^m \tilde{\lambda}_{\dot{\beta}} \mathcal{T}_{\dot{\alpha}\dot{\beta}} \equiv 2D_0. \quad (3.78)$$

To cure this, one adds  $Q_1$  carrying  $\overline{\text{gh}}\# = 1$ . Then,  $\{Q_0 + Q_1, Q_0 + Q_1\}_D = 2(D_0 + \{Q_0, Q_1\}_D) + \{Q_1, Q_1\}_D$ . Thus, to realize the nilpotency at  $\overline{\text{gh}}\# = 0$ , we must require  $D_0 + [\{Q_0, Q_1\}_D]_0 = 0$ , where  $[\{Q_0, Q_1\}_D]_0$  denotes the  $\overline{\text{gh}}\# = 0$  part of  $\{Q_0, Q_1\}_D$ . Such a structure can only be produced when the antighosts in  $Q_1$  get contracted with the ghosts in  $Q_0$  and disappear. With this in mind, one defines the nilpotent operator  $\delta$ , *acting only on the antighosts*, by  $\delta_{\varphi_{a_0}} \equiv G_{a_0}$ , i.e.

$$\delta \tilde{\omega}_A = D_A, \quad \delta \tilde{b} = \mathcal{T}, \quad \delta \tilde{b}_m = \mathcal{T}_m. \quad (3.79)$$

Then, it is easy to see that  $[\{Q_0, Q_1\}_D]_0 = \delta Q_1$  holds and the equation to solve becomes  $\delta Q_1 = -D_0$ . Since  $D_0$  contains no antighosts, the integrability condition  $\delta D_0 = 0$  is trivially satisfied. An obvious solution is  $Q_1 = -\frac{1}{2}(\tilde{\lambda}_{\dot{\alpha}} \tilde{\lambda}_{\dot{\alpha}} \tilde{b} + \tilde{\lambda} \gamma^m \tilde{\lambda} \tilde{b}_m)$ , for which  $Q_0 + Q_1$  takes the usual form of the BRST operator, valid if the algebra is irreducible. For the reducible case at hand, there is an additional solution of the form  $\gamma_{\bar{p}} Z_{\bar{p}}^m \tilde{b}_m$ , where  $\gamma_{\bar{p}}$  are newly introduced coefficient fields, called the ghosts for ghosts<sup>8</sup>. Indeed,  $\delta(\gamma_{\bar{p}} Z_{\bar{p}}^m \tilde{b}_m) = \gamma_{\bar{p}} Z_{\bar{p}}^m \delta \tilde{b}_m = \gamma_{\bar{p}} Z_{\bar{p}}^m \mathcal{T}_m$ , which vanishes due to the reducibility relation (3.76). Thus the solution for  $Q_1$  is

$$Q_1 = -\frac{1}{2} \left( \tilde{\lambda}_{\dot{\alpha}} \tilde{\lambda}_{\dot{\alpha}} \tilde{b} + \tilde{\lambda} \gamma^m \tilde{\lambda} \tilde{b}_m \right) + \gamma_{\bar{p}} Z_{\bar{p}}^m \tilde{b}_m. \quad (3.80)$$

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<sup>8</sup>They are generally denoted by  $\eta_{a_1}$

We now move on to the analysis at  $\overline{\text{gh}}\# = 1$ . The details are more complicated but the basic logic is entirely similar. As the  $\overline{\text{gh}}\# = 0$  part has been removed, we have  $\{Q_0 + Q_1, Q_0 + Q_1\}_D = 2D_1 + (\text{higher order})$ , where  $\overline{\text{gh}}\# = 1$  part  $D_1$  is given by

$$D_1 = \frac{1}{2}\gamma_{\bar{p}}(\tilde{c}\tilde{b}_m\{\mathcal{T}, Z_{\bar{p}}^m\}_D + \tilde{c}_n\tilde{b}_m\{\mathcal{T}_n, Z_{\bar{p}}^m\}_D). \quad (3.81)$$

In obtaining this result, we have used the fact that  $\{D_A, Z_{\bar{p}}^m\}_D = 0$ , following from the invariance of  $Z_{\bar{p}}^m$  under the local fermionic symmetry, and the reducibility relation (3.76). Although the actual computation of the commutators appearing in the above expression is cumbersome and has not yet been performed, we can proceed further by using the general structure of the algebra. Noting that  $\mathcal{T}$  and  $\mathcal{T}_m$  commute among themselves, we get

$$\mathcal{T}_m\{\mathcal{T}, Z_{\bar{p}}^m\}_D = \{\mathcal{T}, \mathcal{T}_m Z_{\bar{p}}^m\}_D = 0, \quad \mathcal{T}_m\{\mathcal{T}_n, Z_{\bar{p}}^m\}_D = \{\mathcal{T}_n, \mathcal{T}_m Z_{\bar{p}}^m\}_D = 0. \quad (3.82)$$

Further, as  $\mathcal{T}_m Z_{\bar{p}}^m = 0$  are the only linear relations among  $\mathcal{T}_m$ , the vanishing relations above imply that  $\{\mathcal{T}, Z_{\bar{p}}^m\}_D$  and  $\{\mathcal{T}_n, Z_{\bar{p}}^m\}_D$  must be linear combinations of  $Z_{\bar{p}}^m$ . So we must have

$$\{\mathcal{T}, Z_{\bar{p}}^m\}_D = A_{\bar{p}}^{\bar{q}} Z_{\bar{q}}^m, \quad \{\mathcal{T}_n, Z_{\bar{p}}^m\}_D = B_{n\bar{p}}^{\bar{q}} Z_{\bar{q}}^m, \quad (3.83)$$

where  $A_{\bar{p}}^{\bar{q}}$  and  $B_{n\bar{p}}^{\bar{q}}$  are some field-dependent coefficients. Substituting them into  $D_1$  it becomes

$$D_1 = \frac{1}{2}\gamma_{\bar{p}}(\tilde{c}A_{\bar{p}}^{\bar{q}} + \tilde{c}_n B_{n\bar{p}}^{\bar{q}})\tilde{b}_m Z_{\bar{q}}^m. \quad (3.84)$$

Just as before, we now introduce  $Q_2$  to kill this contribution. Focusing at the  $\overline{\text{gh}}\# = 1$  piece of  $\{\sum_{n=0}^2 Q_n, \sum_{n=0}^2 Q_n\}_D$ , we obtain the equation  $\delta Q_2 = -D_1$ . The integrability condition  $\delta D_1 = 0$  can be checked using  $\delta\tilde{b}_m = \mathcal{T}_m$  and  $\mathcal{T}_m Z_{\bar{p}}^m = 0$ .  $Q_2$  can then be constructed by introducing the antighost  $\beta_{\bar{p}}$ , which is conjugate to  $\gamma_{\bar{p}}$  and carries  $\overline{\text{gh}}\# = 2$ , with the definition of  $\delta$  on it as  $\delta\beta_{\bar{q}} = \tilde{b}_m Z_{\bar{q}}^m$ . Explicitly it is given by

$$Q_2 = \frac{1}{2}\gamma_{\bar{p}}(\tilde{c}A_{\bar{p}}^{\bar{q}} + \tilde{c}_n B_{n\bar{p}}^{\bar{q}})\beta_{\bar{q}}. \quad (3.85)$$

The general theory [35] guarantees that this process can be continued consistently to higher orders and the final nilpotent BRST operator  $Q = \sum_n Q_n$  is unique up to canonical transformations in the extended phase space. To know at which stage the series actually terminates requires further calculations, which are left for future study.

#### 4. Towards quantization

In this section, to pave the way for future developments, we will analyze the nature of the remaining problems to be solved for the proper quantization of the theory.

### 4.1 “Free field” basis

To perform the quantization of the above system in a useful way, the first question to ask is whether one can find a “free field” basis as in the case of the superstring. As we will see, this turns out to be a non-trivial problem. To clarify the nature of the difficulty, we will first study the case of the ordinary BST formulation without the extra spinor  $\theta_A$  in the SLC gauge, before tackling the case of the double spinor formalism.

Based on the experience with the PS superstring, the best strategy is to begin with the construction of a self-conjugate spinor field  $S_\alpha$  satisfying the canonical Dirac bracket relation

$$\{S_\alpha(\sigma), S_\beta(\sigma')\}_D = i\delta_{\alpha\beta}\delta(\sigma - \sigma') \quad (4.1)$$

from the original spinor  $\tilde{\theta}_\alpha$  obeying

$$\{\tilde{\theta}_\alpha(\sigma), \tilde{\theta}_\beta(\sigma')\}_D = iG_{\alpha\beta}^{-1}\delta(\sigma - \sigma'), \quad (4.2)$$

where  $G_{\alpha\beta}$  is defined in (3.42). In the SLC gauge with  $\theta_A$  set to zero,  $G_{\alpha\beta}$  is simplified considerably and becomes

$$G_{\alpha\beta} = 2(A + B^m\gamma_m)_{\alpha\beta}, \quad (4.3)$$

$$A \equiv k^+, \quad B^m \equiv -\epsilon^{ij}\partial_i x^+ \partial_j x^m. \quad (4.4)$$

Also, the second class constraint  $\tilde{D}_\alpha$  reduces to

$$\tilde{D}_\alpha = \tilde{k}_\alpha - \frac{i}{2}G_{\alpha\beta}\tilde{\theta}_\beta. \quad (4.5)$$

To construct  $S_\alpha$ , we will need the square root of the matrix  $G$ , namely  $V_{\alpha\beta}$  satisfying

$$V_{\alpha\gamma}V_{\gamma\beta} = G_{\alpha\beta}, \quad (4.6)$$

and its inverse. Similarly to the calculation of the inverse of  $G_{\alpha\beta}$  given in (3.47), their explicit forms are easily found as

$$V_{\alpha\beta} = \xi + \xi^{-1}B_m\gamma^m, \quad (4.7)$$

$$V_{\alpha\beta}^{-1} = \frac{1}{2\sqrt{A^2 - B^2}}(\xi - \xi^{-1}B_m\gamma^m), \quad (4.8)$$

$$\xi \equiv \frac{1}{\sqrt{2}}(\sqrt{A+B} + \sqrt{A-B}), \quad B \equiv \sqrt{B^m B_m}. \quad (4.9)$$

Since  $G_{\alpha\beta}$  and  $V_{\alpha\beta}$  do not depend on  $\tilde{\theta}_\alpha$  nor  $x^-$ , we have  $\{G_{\alpha\beta}, V_{\gamma\delta}\}_P = \{\tilde{D}_\alpha, V_{\beta\gamma}\}_P = 0$ . Furthermore  $V_{\alpha\beta}$  is independent of  $\tilde{\theta}_{\dot{\alpha}}$  and  $\tilde{k}_{\dot{\alpha}}$  and hence it Poisson-commutes with  $\tilde{K}_{\dot{\alpha}}$  and  $\tilde{\theta}_{\dot{\alpha}}$ . Together, this implies the anticommutation relations under the Dirac bracket  $\{V_{\alpha\beta}, V_{\gamma\delta}\}_D = \{V_{\alpha\beta}, \tilde{\theta}_\gamma\}_D = 0$ . This then allows us to define  $S_\alpha$  as

$$S_\alpha = V_{\alpha\beta}\tilde{\theta}_\beta. \quad (4.10)$$

Indeed this satisfies  $\{S_\alpha(\sigma), S_\beta(\sigma')\}_D = V_{\alpha\gamma}(\sigma)V_{\beta\delta}(\sigma')\{\tilde{\theta}_\gamma(\sigma), \tilde{\theta}_\delta(\sigma')\}_D = V_{\alpha\gamma}V_{\beta\delta}iG_{\gamma\delta}^{-1} \times \delta(\sigma - \sigma') = i\delta_{\alpha\beta}\delta(\sigma - \sigma')$ , which is the desired canonical relation.

Next we examine the Dirac brackets among the basic bosonic variables  $(x^M, p^M)$ , which we collectively denote as  $f$ . Since in general  $\{f, G_{\alpha\beta}\}_P \neq 0$  and hence  $\{f, \tilde{D}_\alpha\}_P \neq 0$ , they no longer satisfy the canonical relations under the Dirac bracket. Furthermore it is easy to check that  $\{f, S_\alpha\}_D \neq 0$ .

To find new variables which satisfy the canonical form of Dirac brackets, it is useful to examine the structure of  $\{f, S_\alpha\}_D$  in detail. Explicitly we have

$$\{f, S_\alpha\}_D = \{f, S_\alpha\}_P - \{f, \tilde{D}_\beta\}_P \frac{1}{i} G_{\beta\gamma}^{-1} \{\tilde{D}_\gamma, S_\alpha\}_P. \quad (4.11)$$

Substituting the definition of  $S_\alpha$  (4.10) and the form of  $\tilde{D}_\gamma$  given in (4.5), and using the relation  $0 = \{f, 1\}_P = \{f, V\}_P V^{-1} + V\{f, V^{-1}\}_P$ , this can be rewritten as

$$\{f, S_\alpha\}_D = \frac{1}{2} U_{\alpha\beta} S_\beta, \quad (4.12)$$

$$U_{\alpha\beta} = (\{f, V\}_P V^{-1} + \{f, V^{-1}\}_P V)_{\alpha\beta}. \quad (4.13)$$

It is easy to check that  $U$  is antisymmetric and  $\{U, S_\alpha\}_P = 0$ . The equation (4.12) then implies that if we modify  $f$  into

$$\tilde{f} \equiv f - \frac{i}{4} U_{\alpha\beta} S_\alpha S_\beta, \quad (4.14)$$

we can achieve the desired relation  $\{\tilde{f}, S_\alpha\}_D = 0$ .

Using the explicit form of  $V$  and  $V^{-1}$  one can readily evaluate  $U_{\alpha\beta}$  and hence  $\tilde{f}$ . It turned out that  $x^M$  remain intact while the new momenta, to be denoted by  $p^M$ , are given by

$$p^+ = k^+, \quad (4.15)$$

$$p^- = k^- - \frac{i}{4} \partial_i (\hat{U}_{\alpha\beta}^i S_\alpha S_\beta), \quad (4.16)$$

$$p^m = k^m - \frac{i}{4} \partial_i (\hat{U}_{\alpha\beta}^{mi} S_\alpha S_\beta), \quad (4.17)$$

where

$$\hat{U}^i \equiv -\frac{2}{\xi^2 \sqrt{A^2 - B^2}} \chi^{mi} \gamma^{mn} B^n, \quad (4.18)$$

$$\hat{U}^{mi} \equiv -\frac{1}{\xi^2 \sqrt{A^2 - B^2}} \chi^i \gamma^{mn} B^n, \quad (4.19)$$

$$\chi^{mi} \equiv \epsilon^{ij} \partial_j x^m, \quad \chi^i \equiv \epsilon^{ij} \partial_j x^+. \quad (4.20)$$

We note in passing that, since  $x^M$  and  $k^+$  are unchanged,  $V_{\alpha\beta}$  remains unaltered and is independent of  $S_\alpha$ . Therefore, the relation (4.10) can be immediately inverted as  $\tilde{\theta}_\alpha = V_{\alpha\beta}^{-1} S_\beta$ .

It is tedious but straightforward to check the Dirac brackets among  $x^M$  and the new momenta  $p^M$ . The result is quite satisfying: Their brackets are completely canonical. Summarizing, in the case of the conventional BST formulation in the SLC gauge, we have

succeeded in constructing the basis of fields obeying the canonical Dirac bracket relations

$$\{x^M(\sigma), p_N(\sigma')\}_D = i\delta_N^M \delta(\sigma - \sigma'), \quad \{S_\alpha(\sigma), S_\beta(\sigma')\}_D = i\delta_{\alpha\beta} \delta(\sigma - \sigma'), \quad (4.21)$$

$$\text{all others} = 0. \quad (4.22)$$

We now turn to the case of the double spinor formalism. The analysis becomes much more difficult mainly due to the fact that, even in the SLC gauge,  $\tilde{\theta}_\alpha$  remains in  $G_{\alpha\beta}$  in bilinear products with the new spinor  $\theta_A$  and causes  $\{\tilde{D}_\alpha, G_{\beta\gamma}\}_P$  to be non-vanishing. This renders the crucial Dirac bracket relations  $\{V_{\alpha\beta}, V_{\gamma\delta}\}_D = \{V_{\alpha\beta}, \tilde{\theta}_\gamma\}_D = 0$  no longer valid and hence the relation between  $\tilde{\theta}_\alpha$  and  $S_\alpha$  cannot be given simply by  $\tilde{\theta}_\alpha = V_{\alpha\beta}^{-1} S_\beta$ .

For this reason, we have not, unfortunately, been able to find the expression of  $\tilde{\theta}_\alpha$  in terms of the “free field”  $S_\alpha$  in a closed form. However, we shall give below an evidence of the existence of such a “free field” basis by explicitly constructing  $\tilde{\theta}_\alpha$  as a power expansion in  $S_\alpha$  up to  $\mathcal{O}(S^3)$ . As the calculations are quite involved, we will only sketch the procedure and present the result.

Since all the Dirac brackets to appear will be local, i.e. proportional to  $\delta(\sigma - \sigma')$ , we will often omit for simplicity the arguments of the fields  $\sigma$  and  $\sigma'$  as well as the  $\delta(\sigma - \sigma')$  factor. What we wish to do is to construct  $\tilde{\theta}_\alpha$  satisfying  $\{\tilde{\theta}_\alpha, \tilde{\theta}_\beta\}_D = iG_{\alpha\beta}^{-1}$  in powers of the free field  $S_\alpha$  obeying  $\{S_\alpha, S_\beta\}_D = i\delta_{\alpha\beta}$ . Thus we expand  $\tilde{\theta}_\alpha$  as<sup>9</sup>

$$\tilde{\theta}_\alpha = T_{\alpha p}^{(1)} S_p + \frac{1}{2} T_{\alpha [pq]}^{(2)} S_p S_q + \frac{1}{3} T_{\alpha [pqr]}^{(3)} S_p S_q S_r + \dots, \quad (4.23)$$

where the coefficients  $T$ 's, to be determined, are assumed to (anti-)commute with each other and with  $S_\alpha$  under the Dirac bracket and the indices in the bracket [ ] are totally antisymmetrized. In order to compare the right- and the left-hand sides of the equation  $\{\tilde{\theta}_\alpha, \tilde{\theta}_\beta\}_D = iG_{\alpha\beta}^{-1}$ , we need to first expand  $G_{\alpha\beta}^{-1}$  in powers of  $\tilde{\theta}_\alpha$  and then re-express it in powers of  $S_\alpha$  using (4.23) itself. In the SLC gauge,  $G_{\alpha\beta}$  consists of the part  $g_{\alpha\beta}$  independent of  $\tilde{\theta}_\alpha$  and the remaining part  $(G_1)_{\alpha\beta}$  linear in  $\tilde{\theta}_\alpha$  in the following way:

$$G_{\alpha\beta} = g_{\alpha\beta} + (G_1)_{\alpha\beta}, \quad (4.24)$$

$$g_{\alpha\beta} = 2(p^+ + b_m \gamma^m)_{\alpha\beta}, \quad (G_1)_{\alpha\beta} = 2D_{\gamma, \alpha\beta} \tilde{\theta}_\gamma. \quad (4.25)$$

In the above,  $b_m = -\epsilon^{ij} \pi_j^+ \pi_j^m$ , where  $\pi_i^M$  is the  $\tilde{\theta}$ -independent part of  $\Pi_i^M$ . As for  $p^+$  it is a redefinition of the momentum  $k^+$  given by

$$p^+ = k^+ - f^+ - \partial_i (\tilde{\theta}_\gamma f_\gamma^i), \quad (4.26)$$

where  $f^+$  and  $f_\gamma^i$  are quantities independent of  $\tilde{\theta}_\alpha$  which appear in the expression of  $F^+$  in the SLC gauge as  $F^+ = f^+ + \tilde{\theta}_\gamma f_\gamma + \partial_i \tilde{\theta}_\gamma f_\gamma^i$ . We omit their explicit expressions since we will not need them. What is significant is that  $p^+$  can be shown to commute with  $\tilde{D}_\alpha$  under the Dirac bracket. For this reason, we will treat  $p^+$  as a whole and do not count the  $\tilde{\theta}$  in it as one power of  $\tilde{\theta}$ . Lastly,  $D_{\gamma, \alpha\beta}$  is the quantity which will play a central role in the

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<sup>9</sup>In what follows, the Latin indices  $p, q, r$  etc. run over the same range as the Greek indices such as  $\alpha, \beta$ .



following. It appears in the basic Poisson bracket relation<sup>10</sup>

$$\{\tilde{D}_\gamma(\sigma), G_{\alpha\beta}(\sigma')\}_P = 2D_{\gamma,\alpha\beta}\delta(\sigma - \sigma'), \quad (4.27)$$

and is given by

$$D_{\gamma,\alpha\beta} = \phi_\gamma \delta_{\alpha\beta} + \rho_m \gamma_{\alpha\beta}^m, \quad (4.28)$$

$$\phi_\gamma = f_\gamma - \partial_i f_\gamma^i = 2i\epsilon^{ij} \delta_{\gamma\dot{\gamma}} \partial_i \theta_{\dot{\gamma}} \pi_j^+, \quad \rho_m = -2i\epsilon^{ij} (\gamma_m)_{\gamma\dot{\gamma}} \partial_i \theta_{\dot{\gamma}} \pi_j^+. \quad (4.29)$$

Note that it is linear in  $\theta_{\dot{\alpha}}$  and hence vanishes in the ordinary BST formulation. It satisfies the Jacobi identity

$$D_{\gamma,\alpha\beta} + D_{\alpha,\beta\gamma} + D_{\beta,\gamma\alpha} = 0, \quad (4.30)$$

due to the fact that the LHS of (4.27) is proportional to  $\{\tilde{D}_\gamma, \{\tilde{D}_\alpha, \tilde{D}_\beta\}_P\}_P$ .

Now we can describe the procedure to determine the coefficients  $T^{(1)}, T^{(2)}$  and  $T^{(3)}$  in (4.23). Regarding  $G_{\alpha\beta}$  as a matrix,  $G^{-1}$  can be expanded in powers of  $\tilde{\theta}_\alpha$  as  $G^{-1} = g^{-1} - g^{-1}G_1g^{-1} + g^{-1}G_1g^{-1}G_1g^{-1} - \dots$  and further in powers of  $S_\alpha$  using (4.23) for  $\tilde{\theta}_\alpha$  in  $G_1$ . Then, the left- and the right-hand sides of the equation  $\{\tilde{\theta}_\alpha, \tilde{\theta}_\beta\}_D = iG_{\alpha\beta}^{-1}$  become

$$\begin{aligned} \frac{1}{i}LHS &= T_{\alpha p}^{(1)}T_{\beta p}^{(1)} - \left(T_{\alpha p}^{(1)}T_{\beta[p\sigma]}^{(2)}S_\sigma + (\alpha \leftrightarrow \beta)\right) \\ &\quad + \left(T_{\alpha p}^{(1)}T_{\beta[p\rho\sigma]}^{(3)} + T_{\beta p}^{(1)}T_{\alpha[p\rho\sigma]}^{(3)} - T_{\alpha[p\rho]}^{(2)}T_{\beta[p\sigma]}^{(2)}\right)S_\rho S_\sigma + \dots, \end{aligned} \quad (4.31)$$

$$\begin{aligned} \frac{1}{i}RHS &= \left[g^{-1} - 2g^{-1}D^\gamma g^{-1}T_{\gamma\sigma}^{(1)}S_\sigma \right. \\ &\quad \left. - \left(g^{-1}D^\gamma g^{-1}T_{\gamma[\rho\sigma]}^{(2)} + 4g^{-1}D^\gamma g^{-1}D^\delta g^{-1}T_{\gamma\rho}^{(1)}T_{\delta\sigma}^{(1)}\right)S_\rho S_\sigma + \dots\right]_{\alpha\beta}, \end{aligned} \quad (4.32)$$

where  $D^\gamma$  denotes  $D_{\gamma,\alpha\beta}$  regarded as a matrix. Equating them and comparing the coefficients at each order in  $S_\alpha$ , we obtain a set of equations to solve for  $T^{(n)}$ 's.

At the zero-th order, we get  $T_{\alpha p}^{(1)}T_{\beta p}^{(1)} = g_{\alpha\beta}^{-1}$ , which can be solved as

$$T_{\alpha\beta}^{(1)} = v_{\alpha\beta}^{-1}, \quad (4.33)$$

where  $v_{\alpha\beta}^{-1}$  is the ‘‘square-root’’ of  $g_{\alpha\beta}^{-1}$  satisfying  $v_{\alpha\gamma}^{-1}v_{\gamma\beta}^{-1} = g_{\alpha\beta}^{-1}$  and is given by the previous formula (4.8) for  $V_{\alpha\beta}^{-1}$ , with the substitution  $A = p^+, B_m = b_m$ .

At the first order, the equation to solve becomes  $v_{\alpha\rho}^{-1}T_{\beta[\rho\sigma]}^{(2)} + v_{\beta\rho}^{-1}T_{\alpha[\rho\sigma]}^{(2)} = 2(g^{-1}D^\gamma g^{-1})_{\alpha\beta} \times v_{\gamma\sigma}^{-1}$ . After some analysis this can be solved as

$$T_{\alpha[\beta\gamma]}^{(2)} = -\frac{2}{3}(v_{\alpha\kappa}^{-1}\tilde{D}_{\beta,\gamma\kappa} - v_{\alpha\kappa}^{-1}\tilde{D}_{\gamma,\beta\kappa}), \quad (4.34)$$

$$\tilde{D}_{\alpha,\beta\gamma} \equiv v_{\alpha p}^{-1}v_{\beta q}^{-1}v_{\gamma r}^{-1}D_{p,qr}. \quad (4.35)$$

It should be remarked that the solvability of the equation is rather non-trivial due to the required symmetry property of the coefficient  $T_{\alpha[\beta\gamma]}^{(2)}$ . In fact in obtaining (4.34) the use of the Jacobi identity (4.30) was crucial.

<sup>10</sup>It can be derived using the formulas listed in appendix B and a use of a Fierz identity.

At the second order, the equation to solve becomes considerably more complicated. Nevertheless, after a long analysis, it can be solved to determine  $T_{\alpha[\beta\gamma\delta]}^{(3)}$ . We omit the details and summarize the combined result up to  $\mathcal{O}(S^3)$ :

$$\begin{aligned} \tilde{\theta}_\alpha &= v_{\alpha\beta}^{-1} S_\beta - \frac{2}{3} v_{\alpha p}^{-1} \tilde{D}_{\beta,\gamma p} S_\beta S_\gamma \\ &+ \frac{2}{9} v_{\alpha p}^{-1} (\tilde{D}_{p,\gamma q} \tilde{D}_{\beta,\delta q} - 5 \tilde{D}_{\gamma,pq} \tilde{D}_{\delta,\beta q}) S_\beta S_\gamma S_\delta + \mathcal{O}(S^4). \end{aligned} \quad (4.36)$$

Since  $S_\alpha$  is fermionic, this series will terminate at  $\mathcal{O}(S^{16})$  at the worst. Although it is difficult to guess the closed form expression at the present time, the result above strongly suggests the existence of the “free field” basis for the double spinor formalism as well.

## 4.2 Requirements for proper quantization

In the above, we have shown that in the SLC gauge a basis of fields in which the equal-time Dirac brackets among them take the canonical “free field” form can be constructed in closed form for the usual BST formulation and gave an evidence that it also exists in the double spinor formalism. Assuming that it exists, the usual step for the quantization of the system is to replace the Dirac brackets by the quantum (anti-)commutators. It is important, however, to recognize that this procedure constitutes only a part of the quantization. Below we discuss the requirements for a proper quantization and briefly explore a possible scheme to realize them for the supermembrane case.

Logically, a classical theory does not determine a corresponding quantum system. In addition to giving the equal-time (anti-)commutation relations for the basic fields, one must also specify (i) the operator products between all the fundamental variables including the nature of short-distance singularities and (ii) the way to define the composite operators which have finite matrix elements. These specifications themselves must satisfy certain requirements. The most important among them is that they retain the local symmetries governing the degrees of freedom of the system. Otherwise anomalies may result and the quantum system becomes inconsistent. Another requirement is that the physical observables should be hermitian. Further, one often demands that the global symmetries of the classical theory remain intact. It is not known, however, whether these requirements uniquely fix the quantum theory. There may exist more than one set of rules which satisfy all the requirements. In such a case, they define different but consistent quantum extensions of a given classical theory and a choice among them can only be decided by physical experiments.

In the case where the theory can be treated perturbatively starting from a free Lagrangian, the procedure of constructing a consistent quantum theory from a classical theory has been systematized through many efforts over the years and is now regarded as a textbook matter. It is instructive, however, to recall the reason why this has been possible. For a genuine free theory, one not only has the canonical equal time commutation relations between the conjugate fields, such as  $[\phi(t, \vec{x}), \Pi_\phi(t, \vec{x}')] = i\delta(\vec{x} - \vec{x}')$ , but also the crucial relations between them, like  $\Pi_\phi = \partial_t \phi$ . Because of this, the operator products (or the basic correlation functions) among the free fields are completely fixed including the singularities

at the coincident point. This then allows one to easily define finite composite operators by removing such singularities. When one turns on the interactions, infinities arise at the loop level. But since their structures follow from prescribed rules, one can find suitable symmetry-preserving regularization scheme and perform renormalization in a systematic manner.

In contrast, the system that we are dealing with does not admit such a perturbative treatment. In fact the information on the dynamics is contained in its entirety in the set of first class constraints in the phase space. Even if one finds a basis of fields where conjugate fields enjoy canonical (anti-)commutation relations, such as  $\{x^M(t, \sigma), p_N(t, \sigma')\}_D = \delta_N^M \delta(\sigma - \sigma')$  or its quantum replacement, they are not related simply, like  $p^M = \partial_t x^M$ , reflecting the fact that they need not be bonafide free fields. In this situation, what is crucial is to define the complete operator products of these fields and the prescription to render the composite operators finite *in such a way that the the constraint algebra is realized quantum mechanically*. In the case of the superstring studied in [22], it turned out that this was achieved simply by assuming the free-field operator product together with the usual radial normal-ordering and adding a few quantum improvement terms to the constraints. In this way the free-field postulate of Berkovits was fully justified. In the case of the supermembrane, however, it is not obvious that a similar prescription will work. We must postulate certain rules and see if they lead to a consistent quantum theory.

Although this problem has not been solved, we shall present a preliminary investigation, which helps clarify the nature of the problem. In what follows, we will concentrate on the local consistency and ignore possible global issues. Specifically, we will consider the case where the worldvolume is of the structure  $R \times \Sigma$ , where  $R$  denotes the non-compact time-like direction and  $\Sigma$  is a compact spatial 2-surface admitting a real complete orthonormal basis  $\{Y_I(\sigma)\}$  for the functions on  $\Sigma$  with the properties

$$\int d^2\sigma Y_I(\sigma)Y_J(\sigma) = \delta_{IJ}, \quad \sum_I Y_I(\sigma)Y_I(\sigma') = \delta(\sigma - \sigma'). \quad (4.37)$$

Consider first the bosonic field  $x^M(t, \sigma)$  and its conjugate  $p^N(t, \sigma)$ , which are assumed to satisfy the canonical commutation relations

$$[x^M(t, \sigma), p^N(t, \sigma')] = i\eta^{MN} \delta(\sigma - \sigma'), \quad (4.38)$$

$$[x^M(t, \sigma), x^N(t, \sigma')] = [p^M(t, \sigma), p^N(t, \sigma')] = 0. \quad (4.39)$$

We expand them in the basis above as

$$x^M(t, \sigma) = \sum_I x_I^M(t) Y_I(\sigma), \quad p^N(t, \sigma) = \sum_J p_J^N(t) Y_J(\sigma), \quad (4.40)$$

and further decompose  $x^M(t)$  and  $p^M(t)$  into positive and negative frequency parts in the following way:

$$x_I^M(t) = \frac{1}{\sqrt{2\Omega}} \int_0^\Omega d\omega \left( a_I^M(\omega) e^{i\omega t} + a_I^{\dagger M}(\omega) e^{-i\omega t} \right), \quad (4.41)$$

$$p_I^M(t) = \frac{1}{\sqrt{2\Omega}} \int_0^\Omega d\omega \left( b_I^M(\omega) e^{i\omega t} + b_I^{\dagger M}(\omega) e^{-i\omega t} \right). \quad (4.42)$$

Here,  $\Omega$  is a cutoff in frequency to make subsequent computations well-defined. It is easy to see that to realize (4.38) we must set

$$\left[ a_I^M(\omega), b_J^{\dagger N}(\omega') \right] = \left[ a_I^{\dagger M}(\omega), b_J^N(\omega') \right] = i\eta^{MN} \delta_{IJ} \delta(\omega - \omega'), \quad (4.43)$$

$$\left[ a_I^M(\omega), b_J^N(\omega') \right] = \left[ a_I^{\dagger M}(\omega), b_J^{\dagger N}(\omega') \right] = 0. \quad (4.44)$$

Note that the commutators  $[a, a^\dagger]$ ,  $[a, a]$ ,  $[b, b^\dagger]$  and  $[b, b]$  are not yet determined because of the lack of explicit relation between  $x^M$  and  $p^M$ . Now further imposing (4.39) we get

$$\left[ a_I^M(\omega), a_J^{\dagger N}(\omega') \right] = \eta^{MN} f_a(I, J) \delta(\omega - \omega'), \quad (4.45)$$

$$\left[ b_I^M(\omega), b_J^{\dagger N}(\omega') \right] = \eta^{MN} f_b(I, J) \delta(\omega - \omega'), \quad (4.46)$$

$$[a, a] = [b, b] = 0, \quad (4.47)$$

where the only restrictions on  $f_a(I, J)$  and  $f_b(I, J)$  are that they are hermitian and symmetric in  $I, J$ . Generically they may even depend on the operators  $a, a^\dagger, b, b^\dagger$ . With respect to the states built on the Fock vacuum  $|0\rangle$  characterized by  $a|0\rangle = b|0\rangle = 0$ , one can define finite normal-ordered product of two operators  $:AB:$  in the usual way, namely by placing  $a$  and  $b$  to the right of  $a^\dagger$  and  $b^\dagger$ . Then the product of spatially separated operators can be written as

$$A(t, \sigma)B(t, \sigma') = \underbrace{A(t, \sigma)B(t, \sigma')} + :A(t, \sigma)B(t, \sigma'):, \quad (4.48)$$

which should be regarded as the definition of the ‘‘contraction’’  $\underbrace{A(t, \sigma)B(t, \sigma')}$ . Applied to  $x^M$  and  $p^M$  in question, we readily get

$$\underbrace{x^M(t, \sigma)p^N(t, \sigma')} = \frac{i}{2} \eta^{MN} \delta(\sigma - \sigma') = \frac{1}{2} [x^M(t, \sigma), p^N(t, \sigma')], \quad (4.49)$$

$$\underbrace{x^M(t, \sigma)x^N(t, \sigma')} = \frac{1}{2} \eta^{MN} \sum_{I, J} f_a(I, J) Y_I(\sigma) Y_J(\sigma'), \quad (4.50)$$

$$\underbrace{p^M(t, \sigma)p^N(t, \sigma')} = \frac{1}{2} \eta^{MN} \sum_{I, J} f_b(I, J) Y_I(\sigma) Y_J(\sigma'). \quad (4.51)$$

This makes it crystal clear that, while the singularity of the product of mutually conjugate fields is canonical, the one between the non-conjugates is dictated by the functions  $f_a(I, J)$  and  $f_b(I, J)$ . The main task of the proper quantization is to choose these functions so that the algebra of first class constraints (with possible modifications of their explicit forms) is maintained quantum mechanically.

Although this is a difficult problem, a progress can be made if a solution exists within the assumption that these functions can be chosen to be  $c$ -numbers. In this case, the normal-ordered product for more than two fields can similarly be defined by pushing the

annihilation operators to the right. This gives the familiar recursive definitions (for bosonic fields)

$$A : B_1 B_2 \cdots B_n : =: AB_1 B_2 \cdots B_n : + \sum_{i=1}^n \underbrace{A}_{\square} B_i : B_1 \cdots \check{B}_i \cdots B_n : , \quad (4.52)$$

where  $\check{B}_i$  signifies the removal of  $B_i$ . In this form the fact that the normal ordering removes all possible singularities is manifest. Furthermore, it can be shown, again recursively, that the normal-ordered product of any number of hermitian fields is hermitian. Therefore, once the construction of the canonical basis of fields is completed, it should be possible to examine the quantum algebra of constraints just as in the case of genuine free fields. In fact the above  $c$ -number assumption appears reasonable from the point of view of bose-fermi symmetry. If we repeat the analysis given above for the canonical fermionic field  $S_\alpha$  satisfying  $\{S_\alpha(t, \sigma), S_\beta(t, \sigma')\} = \delta_{\alpha\beta} \delta(\sigma - \sigma')$ , we easily find that the singularity in the operator product must be canonical, namely  $\underbrace{S_\alpha(t, \sigma)}_{\square} S_\beta(t, \sigma') = \frac{1}{2} \delta_{\alpha\beta} \delta(\sigma - \sigma')$ . This is due to the self-conjugate nature of  $S_\alpha$ . Now if the local singularity structure is not drastically altered in the double spinor formalism compared to the usual BST formalism,  $S_\alpha$  and  $x^M$  should be related by supersymmetry. Then, it would be rather unnatural if the singularity in  $x^M(t, \sigma)x^N(t, \sigma')$  is operator valued while the one for  $S_\alpha(t, \sigma)S_\beta(t, \sigma')$  is a  $c$ -number. In any case, whether the quantum constraint algebra can be realized with the assumption above should be examined carefully in a future work.

## 5. Summary and discussions

In this work we started an attempt towards pure spinor type covariant quantization of the supermembrane in 11 dimensions as an application of the “double spinor” formalism that we developed previously for the superstring. Starting from a simple generalization of the conventional BST action with doubled spinor degrees of freedom and a new local fermionic symmetry, we carefully analyzed the structure of the constraints in the semi-light-cone gauge. Although the amount of computation is far greater than in the string case, in the end we found a very simple algebra of first class constraints which governs the entire dynamics of the theory. This demonstrates that at least at the classical level the double spinor formalism works nicely for the supermembrane as well. In order to quantize the theory in a tractable way, one must find the basis in which the fundamental fields obey canonical Dirac bracket relations. For the BST formulation in the SLC gauge we were able to construct such a basis in closed form, while for the double spinor formulation we indicated its existence by constructing it in the power series in the canonical fermionic field. Finally we discussed in some detail what are required for the proper quantization and suggested a direction to pursue.

Although the fact that the structure of the all-important first class algebra remained simple even for the supermembrane is remarkable and encouraging, clearly much work is needed for its quantization and the subsequent extraction of the “covariant core” of the BRST operator. Besides further developing the type of analysis presented in section 4, it

would be interesting and instructive to study in detail how our results reduce to the case of type IIA superstring upon appropriate dimensional reduction. We hope to report on these and related matters elsewhere.

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## A. Notations and conventions

We take the 32 dimensional Majorana representation for the  $SO(10,1)$  gamma matrices  $\Gamma^M$  and they satisfy

$$\{\Gamma^M, \Gamma^N\} = 2\eta^{MN}. \quad (\text{A.1})$$

Our convention for the metric is mostly plus:  $\eta = (-, +, \dots, +)$ . The lightcone decomposition of a vector  $v^M$  is taken as

$$v^M = (v^\pm, v^m), \quad v^\pm = v^0 \pm v^{10}, \quad m = 1, \dots, 9. \quad (\text{A.2})$$

We also define  $\Gamma^\pm = \Gamma^0 \pm \Gamma^{10}$ . The lightcone chirality operator (i.e. the  $SO(1,1)$  boost charge) is defined as  $\hat{\Gamma} \equiv \Gamma^0 \Gamma^{10}$  and we take a basis of spinors in which  $\hat{\Gamma}$  is diagonal. A 32 dimensional spinor  $\phi_A$  is decomposed as  $\phi_A = (\phi_\alpha, \phi_{\dot{\alpha}})$  ( $\alpha, \dot{\alpha} = 1, \dots, 16$ ) according to the  $\hat{\Gamma}$ -eigenvalue ( $\hat{\Gamma}_{\alpha\beta}\phi_\beta = \phi_\alpha$  and  $\hat{\Gamma}_{\dot{\alpha}\dot{\beta}}\phi_{\dot{\beta}} = -\phi_{\dot{\alpha}}$ ), and explicitly, we have

$$\Gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Gamma^m = \begin{pmatrix} -\gamma^m & 0 \\ 0 & \gamma^m \end{pmatrix}, \quad \Gamma^{10} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\Gamma} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{A.3})$$

Note that this basis differs from the ten dimensional chirality basis in which  $\Gamma^{10}$  is diagonal. The charge conjugation matrix  $C$  is characterized by the property  $C(\Gamma^\mu)^T C^{-1} = \Gamma^\mu$  and it coincides with  $\Gamma^0$ . Some formulas useful to remember are

$$C\Gamma^+ = \begin{pmatrix} 0 & 0 \\ 0 & -2\delta_{\dot{\alpha}\dot{\beta}} \end{pmatrix}, \quad C\Gamma^- = \begin{pmatrix} -2\delta_{\alpha\beta} & 0 \\ 0 & 0 \end{pmatrix}, \quad C\Gamma^{+-} = \begin{pmatrix} 0 & 2\delta_{\alpha\dot{\beta}} \\ 2\delta_{\dot{\alpha}\beta} & 0 \end{pmatrix}, \quad (\text{A.4})$$

$$C\Gamma^{+m} = \begin{pmatrix} 0 & 0 \\ 0 & -2\gamma_{\dot{\alpha}\dot{\beta}}^m \end{pmatrix}, \quad C\Gamma^{-m} = \begin{pmatrix} 2\gamma_{\alpha\beta}^m & 0 \\ 0 & 0 \end{pmatrix}, \quad C\Gamma^{mn} = \begin{pmatrix} 0 & \gamma_{\alpha\dot{\beta}}^{mn} \\ \gamma_{\dot{\alpha}\beta}^{mn} & 0 \end{pmatrix}. \quad (\text{A.5})$$

The indices of the  $16 \times 16$  components have the following symmetries

$$\gamma_{\alpha\beta}^m = \gamma_{\beta\alpha}^m, \quad \gamma_{\dot{\alpha}\dot{\beta}}^m = \gamma_{\dot{\beta}\dot{\alpha}}^m, \quad \gamma_{\alpha\dot{\beta}}^{mn} = -\gamma_{\dot{\beta}\alpha}^{mn}, \quad (\text{A.6})$$

and  $\gamma_{\alpha\beta}^m$  and  $\gamma_{\dot{\alpha}\dot{\beta}}^m$  obey the  $SO(9)$  Clifford algebra.

Finally, we note the basic Fierz identity in 11 dimensions, which we utilized in many of the calculations in the main text:

$$0 = (C\Gamma^M)_{AB}(C\Gamma_{MN})_{CD} + (C\Gamma^M)_{AC}(C\Gamma_{MN})_{DB} + (C\Gamma^M)_{AD}(C\Gamma_{MN})_{BC} \\ + (C\Gamma_{MN})_{AB}(C\Gamma_M)_{CD} + (C\Gamma_{MN})_{AC}(C\Gamma_M)_{DB} + (C\Gamma_{MN})_{AD}(C\Gamma_M)_{BC}. \quad (\text{A.7})$$

## B. List of useful formulas in the SLC gauge

In order to keep the length of the paper reasonable, we had to omit many of the calculations, especially those performed in the SLC gauge, where  $\tilde{\theta}_\alpha = 0$ . To partially compensate this omission, below we will collect some useful formulas for the basic quantities in this gauge.

First, before imposing the gauge condition, the basic building blocks  $\Pi_i^M, W_i^M(\Theta)$  and  $W_i^{MN}(\Theta)$  are decomposed as

$$\Pi_i^M = \partial_i x^M - i\partial_i(\theta C\Gamma^M \tilde{\theta}) - W_i^M(\Theta) = \pi_i^M - \tilde{w}_i^M + 2\tilde{a}_i^M, \quad (\text{B.1})$$

$$W_i^M(\Theta) = i\Theta C\Gamma^M \partial_i \Theta = w_i^M + \tilde{w}_i^M - (a_i^M + \tilde{a}_i^M), \quad (\text{B.2})$$

$$W_i^{MN}(\Theta) = i\Theta C\Gamma^{MN} \partial_i \Theta = w_i^{MN} + \tilde{w}_i^{MN} - (b_i^{MN} + \tilde{b}_i^{MN}), \quad (\text{B.3})$$

where the variables in lowercase letter are given by

$$\pi_i^M = \partial_i x^M - w_i^M, \quad (\text{B.4})$$

$$w_i^M = W_i^M(\theta), \quad \tilde{w}_i^M = W_i^M(\tilde{\theta}), \quad (\text{B.5})$$

$$a_i^M = i\theta C\Gamma^M \partial_i \tilde{\theta}, \quad \tilde{a}_i^M = i\tilde{\theta} C\Gamma^M \partial_i \theta, \quad (\text{B.6})$$

$$w_i^{MN} = W_i^{MN}(\theta), \quad \tilde{w}_i^{MN} = W_i^{MN}(\tilde{\theta}) \quad (\text{B.7})$$

$$b_i^{MN} = i\theta C\Gamma^{MN} \partial_i \tilde{\theta}, \quad \tilde{b}_i^{MN} = i\tilde{\theta} C\Gamma^{MN} \partial_i \theta. \quad (\text{B.8})$$

$a_i^M, \tilde{a}_i^M, b_i^{MN}$  and  $\tilde{b}_i^{MN}$  are linear in  $\tilde{\theta}$  and  $\theta$ .

Now we go to the SLC gauge by setting  $\tilde{\theta}_\alpha = 0$ . In the following, we will use the notation  $\hat{\theta} \equiv (\theta_\alpha), \theta \equiv \theta_\alpha, \tilde{\theta} \equiv (\tilde{\theta}_\alpha)$ . Then the non-vanishing light-cone components of the variables in lowercase letter are

$$\tilde{w}_i^- = -2i\tilde{\theta}\partial_i\tilde{\theta}, \quad \tilde{w}_i^{-m} = 2i\tilde{\theta}\gamma^m\partial_i\tilde{\theta}, \quad (\text{B.9})$$

$$a_i^m = i\hat{\theta}\gamma^m\partial_i\tilde{\theta}, \quad a_i^- = -2i\theta\partial_i\tilde{\theta}, \quad (\text{B.10})$$

$$\tilde{a}_i^m = i\tilde{\theta}\gamma^m\partial_i\hat{\theta}, \quad \tilde{a}_i^- = -2i\tilde{\theta}\partial_i\theta, \quad (\text{B.11})$$

$$b_i^{+-} = 2i\hat{\theta}\partial_i\tilde{\theta}, \quad b_i^{-m} = 2i\theta\gamma^m\partial_i\tilde{\theta}, \quad b_i^{mn} = -i\hat{\theta}\gamma^{mn}\partial_i\tilde{\theta}, \quad (\text{B.12})$$

$$\tilde{b}_i^{+-} = 2i\tilde{\theta}\partial_i\hat{\theta}, \quad \tilde{b}_i^{-m} = 2i\tilde{\theta}\gamma^m\partial_i\theta, \quad \tilde{b}_i^{mn} = -i\tilde{\theta}\gamma^{mn}\partial_i\hat{\theta}. \quad (\text{B.13})$$

In terms of them, the light-cone components of the basic building blocks become

$$\Pi_i^+ = \pi_i^+, \quad \Pi_i^- = \pi_i^- - \tilde{w}_i^- + 2\tilde{a}_i^-, \quad \Pi_i^m = \pi_i^m + 2\tilde{a}_i^m, \quad (\text{B.14})$$

$$W_i^+ = w_i^+, \quad W_i^- = w_i^- + \tilde{w}_i^- - (a_i^- + \tilde{a}_i^-), \quad W_i^m = w_i^m - (a_i^m + \tilde{a}_i^m), \quad (\text{B.15})$$

$$W_i^{+-} = w_i^{+-} - (b_i^{+-} + \tilde{b}_i^{+-}), \quad W_i^{+m} = w_i^{+m}, \quad (\text{B.16})$$

$$W_i^{-m} = w_i^{-m} + \tilde{w}_i^{-m} - (b_i^{-m} + \tilde{b}_i^{-m}), \quad W_i^{mn} = w_i^{mn} - (b_i^{mn} + \tilde{b}_i^{mn}). \quad (\text{B.17})$$

The quantities which play important roles in the text are  $G_{\alpha\beta}$  and  $\tilde{D}_\alpha$  given by

$$G_{\alpha\beta} = 2(A + B_m \gamma^m)_{\alpha\beta}, \quad (\text{B.18})$$

$$A = k^+ - F^+, \quad B_m = -\epsilon^{ij} \Pi_i^+ \Pi_{mj}, \quad (\text{B.19})$$

$$\tilde{D}_\alpha = \tilde{k}_\alpha - ik^+ \tilde{\theta}_\alpha + \tilde{\Phi}_\alpha. \quad (\text{B.20})$$

In the SLC gauge,  $B_m$  simplifies to  $B_m = -\epsilon^{ij} \pi_i^+ (\pi_j^m + 2\tilde{a}_j^m)$ .  $F^+$  becomes

$$F^+ = F^{(0)+} + F^{(1)+}, \quad (\text{B.21})$$

$$F^{(0)+} = -\frac{1}{2} \epsilon^{ij} w_i^{+-} \left( \pi_j^+ + \frac{1}{2} w_j^+ \right) + \epsilon^{ij} w_i^{+m} \left( \pi_j^m + \frac{1}{2} w_j^m \right), \quad (\text{B.22})$$

$$F^{(1)+} = \frac{1}{2} \epsilon^{ij} (b_i^{+-} + \tilde{b}_i^{+-}) \left( \pi_j^+ + \frac{1}{2} w_j^+ \right) + \frac{1}{2} \epsilon^{ij} w_i^{+m} (3\tilde{a}_j^m - a_j^m), \quad (\text{B.23})$$

where  $F^{(0)+}$  is independent of  $\tilde{\theta}$  and  $F^{(1)+}$  is linear in  $\tilde{\theta}$ .  $\tilde{\Phi}_\alpha$  is still complicated. If we split it as

$$\tilde{\Phi}_\alpha = \tilde{\Phi}_\alpha^{(A)} + \tilde{\Phi}_\alpha^{(B)}, \quad (\text{B.24})$$

$$\tilde{\Phi}_\alpha^{(A)} = -\frac{i}{2} \epsilon^{ij} (\bar{\Theta} \Gamma^{MN})_\alpha \left( \Pi_{Mi} \Pi_{Nj} + \Pi_{Mi} W_{Nj} + \frac{1}{3} W_{Mi} W_{Nj} \right), \quad (\text{B.25})$$

$$\tilde{\Phi}_\alpha^{(B)} = \frac{i}{2} \epsilon^{ij} W_i^{MN} (\bar{\Theta} \Gamma_M)_\alpha \left( \Pi_{Nj} + \frac{2}{3} W_{Nj} \right), \quad (\text{B.26})$$

the light-cone decompositions for  $\tilde{\Phi}_\alpha^{(A)}$  and  $\tilde{\Phi}_\alpha^{(B)}$  take the form

$$\begin{aligned} \tilde{\Phi}_\alpha^{(A)} = & -\frac{i}{2} \epsilon^{ij} \left\{ -\frac{1}{2} (\acute{\theta}_\beta \delta_{\beta\alpha}) \left( 2\Pi_i^- \pi_j^+ + \Pi_i^- w_j^+ - \pi_i^+ W_j^- + \frac{2}{3} W_i^- w_j^+ \right) \right. \\ & + (\Theta \gamma^m)_\alpha \left( 2\Pi_i^m \pi_j^+ + \Pi_i^m w_j^+ - \pi_i^+ W_j^m + \frac{2}{3} W_i^m w_j^+ \right) \\ & \left. + (\acute{\theta} \gamma^{mn})_\alpha \left( \Pi_i^m \Pi_j^n + \Pi_i^m W_j^n + \frac{1}{3} W_i^m W_j^n \right) \right\}, \end{aligned} \quad (\text{B.27})$$

$$\begin{aligned} \tilde{\Phi}_\alpha^{(B)} = & \frac{i}{2} \epsilon^{ij} \left\{ -\frac{1}{2} \left( \Theta_\alpha W_i^{+-} + (\acute{\theta} \gamma_m)_\alpha W_i^{-m} \right) \left( \pi_j^+ + \frac{2}{3} w_j^+ \right) \right. \\ & + \left( \Theta_\alpha w_i^{+n} - (\acute{\theta} \gamma_m)_\alpha W_i^{mn} \right) \left( \Pi_j^n + \frac{2}{3} W_j^n \right) \\ & \left. - \frac{1}{2} (\acute{\theta} \gamma_m)_\alpha w_i^{+m} \left( \Pi_j^- + \frac{2}{3} W_j^- \right) \right\}. \end{aligned} \quad (\text{B.28})$$

### C. Equivalence between $\{\mathcal{T}, \mathcal{T}_m\}$ and $\{T^{(0)}, T_i^{(1)}\}$

In this appendix, we will show that under generic conditions the 10 constraints  $\{\mathcal{T}, \mathcal{T}_m\}$  are equivalent to the original bosonic constraints  $\{T^{(0)}, T_i^{(1)}\}$ .



$\mathcal{T}$  and  $\mathcal{T}_m$ , given in (3.52) and (3.53), can be conveniently written as

$$\mathcal{T} = \frac{K}{2i(A^2 - B^2)}, \quad \mathcal{T}_m = \frac{K_m}{2i(A^2 - B^2)}, \quad (\text{C.1})$$

$$K = a^i T_i^{(1)} + bT^{(0)}, \quad K_m = c_m^i T_i^{(1)} + d_m T^{(0)}, \quad (\text{C.2})$$

where

$$A = \mathcal{K}^+, \quad B_m = -\epsilon^{ij} \Pi_i^+ \Pi_{mj}, \quad B^2 = B_m B^m, \quad (\text{C.3})$$

$$a^i = 4\epsilon^{ij} \Pi_j^+ (\epsilon^{kl} \Pi_k^- \Pi_l^+) - 8B_m \epsilon^{ij} \Pi_j^m, \quad b = -4A, \quad (\text{C.4})$$

$$c_m^i = 8\epsilon^{ij} (\Pi_j^+ \mathcal{K}_m - \mathcal{K}^+ \Pi_{mj}), \quad d_m = -4B_m, \quad (\text{C.5})$$

and  $\mathcal{K}^M$  was given in (3.28). As stated previously, we assume  $A^2 - B^2 \neq 0$  and  $A \neq 0$ . Since  $T^{(1)} = T_i^{(2)} = 0 \Rightarrow \mathcal{T} = \mathcal{T}_m = 0$  is trivial, what should be shown is the converse.

First from the form of  $c_m^i$  and the definition of  $B_m$ , one finds that  $\Pi_i^+ c_m^i = 8AB_m$  and hence  $d_m = -4B_m = -(1/2A)\Pi_i^+ c_m^i$ . Putting this into  $K_m$  yields

$$2i(A^2 - B^2)\mathcal{T}_m = c_m^i \left( T_i^{(1)} - \frac{\Pi_i^+}{2A} T^{(0)} \right). \quad (\text{C.6})$$

Now generically the  $2 \times 9$  matrix  $c_m^i$  has rank 2 and  $C^{ij} \equiv c_m^i c_m^j$  is invertible. This means that the linear combination  $2i(A^2 - B^2)C^{-1}{}_{ij} c_m^j \mathcal{T}_m$  is actually equal to  $T_i^{(1)} - (1/2A)\Pi_i^+ T^{(0)}$ . Thus  $\mathcal{T}_m = 0$  implies the following relation

$$T_i^{(1)} = \frac{\Pi_i^+}{2A} T^{(0)}. \quad (\text{C.7})$$

Next we note the simple relation  $\Pi_i^+ a^i = 8B^2$ , which can be easily checked. We can use this relation and (C.7) to rewrite  $K$  into the form

$$K = a^i T_i^{(1)} + bT^{(0)} = -\frac{4}{A}(A^2 - B^2)T^{(0)}. \quad (\text{C.8})$$

Setting this to zero yields  $T^{(0)} = 0$  and this in turn gives  $T_i^{(1)} = 0$  from (C.7). Therefore under generic conditions the constraints  $\mathcal{T} = 0, \mathcal{T}_m = 0$  are equivalent to the original constraints  $T^{(0)} = 0, T_i^{(1)} = 0$ .

#### D. First order reducibility function $Z_{\bar{p}}^{\bar{m}}$

In this appendix, we give the explicit form of the first order reducibility function  $Z_{\bar{p}}^{\bar{m}}$  satisfying

$$Z_{\bar{p}}^{\bar{m}} \mathcal{T}_{\bar{m}} = 0, \quad (\text{D.1})$$

where  $\mathcal{T}_{\bar{m}} = (\mathcal{T}_0 = \mathcal{T}, \mathcal{T}_m)$ . From the explicit form of  $\mathcal{T}_{\bar{m}}$  given in the appendix C, it is easy to see that (D.1) above is equivalent to

$$AZ_{\bar{p}}^0 + B_m Z_{\bar{p}}^m = 0, \quad (\text{D.2})$$

$$a^i Z_{\bar{p}}^0 + c_m^i Z_{\bar{p}}^m = 0. \quad (\text{D.3})$$

First contract (D.3) with  $\Pi_i^+$ . Using the relation  $\Pi_i^+ a^i = 8B^2$  already utilized in the appendix C and a similar one  $\Pi_i^+ c_m^i = 8AB_m$ , which can be easily checked, we get

$$B^2 Z_{\bar{p}}^0 + AB_m Z_{\bar{p}}^m = 0. \quad (\text{D.4})$$

Combining (D.2) and (D.4) we deduce

$$0 = \frac{A^2 - B^2}{AB^2} B_m Z_{\bar{p}}^m. \quad (\text{D.5})$$

Since  $A^2 - B^2 \neq 0$ , this implies  $B_m Z_{\bar{p}}^m = 0$  and further from (D.2) we get  $Z_{\bar{p}}^0 = 0$ .

To solve  $B_m Z_{\bar{p}}^m = 0$  for  $Z_{\bar{p}}^m$  explicitly, recall that  $B_m$  is given by  $B_m = -\epsilon^{ij} \Pi_i^+ \Pi_{mj}$ . Thus the equation can be written as

$$\epsilon^{ij} \Pi_i^+ Y_{j\bar{p}} = 0, \quad (\text{D.6})$$

where  $Y_{j\bar{p}} \equiv \Pi_{mj} Z_{\bar{p}}^m$ . Its general solution is given by

$$Y_{j\bar{p}} = C_{\bar{p}} \Pi_j^+, \quad C_{\bar{p}} = \text{arbitrary}. \quad (\text{D.7})$$

Now if we employ the 9-vector notation  $\vec{Z}_{\bar{p}} = (Z_{\bar{p}}^m)$ , the relation  $Y_{j\bar{p}} = \Pi_{mj} Z_{\bar{p}}^m$  can be written as

$$\vec{Z}_{\bar{p}} \cdot \vec{\Pi}_1 = \Pi_1^+, \quad \vec{Z}_{\bar{p}} \cdot \vec{\Pi}_2 = \Pi_2^+. \quad (\text{D.8})$$

where, without loss of generality, we have absorbed the factor  $C_{\bar{p}}$  into  $\vec{Z}_{\bar{p}}$ . Since there are precisely 7 vectors orthogonal to the plane spanned by  $\vec{\Pi}_i$ , we will denote them by  $\vec{X}_{\bar{p}}$ . Then the general solution to (D.8) is of the form  $\vec{Z}_{\bar{p}} = \vec{W} + \vec{X}_{\bar{p}}$ , where  $\vec{W}$  is the unique solution of (D.8) lying in the  $\vec{\Pi}_1$ - $\vec{\Pi}_2$  plane. Making the expansion  $\vec{W} = \xi_i \vec{\Pi}_i$  and plugging it into (D.8), the equation for the coefficients becomes  $R_{ij} \xi_j = \Pi_i^+$ , where  $R_{ij} \equiv \vec{\Pi}_i \cdot \vec{\Pi}_j = \Pi_i^m \Pi_j^m$ . The solution is

$$\xi_i = R_{ij}^{-1} \Pi_j^+, \quad R_{ij}^{-1} = \frac{\epsilon_{ik} \epsilon_{jl} R_{kl}}{\det R}. \quad (\text{D.9})$$

Thus the complete solution for  $Z_{\bar{p}}^m$ , up to an overall factor for each  $\bar{p}$ , is given by

$$\vec{Z}_{\bar{p}} = \vec{X}_{\bar{p}} + R_{ij}^{-1} \Pi_j^+ \vec{\Pi}_i, \quad \vec{X}_{\bar{p}} \cdot \vec{\Pi}_i = 0 \quad (\text{D.10})$$

$$Z_{\bar{p}}^0 = 0. \quad (\text{D.11})$$

It is easy to see that the reducibility is of first order, i.e. there are no further linear relations among the seven vectors  $\vec{Z}_{\bar{p}}$ . Indeed  $0 = \lambda^{\bar{p}} \vec{Z}_{\bar{p}} = \lambda^{\bar{p}} \vec{X}_{\bar{p}} + (\sum_{\bar{p}} \lambda^{\bar{p}}) R_{ij}^{-1} \Pi_j^+ \vec{\Pi}_i$  implies  $\lambda^{\bar{p}} = 0$ , since  $\vec{X}_{\bar{p}}$  are orthogonal to  $\vec{\Pi}_i$  and are linearly independent.

## References

- [1] N. Berkovits, *Super-Poincaré covariant quantization of the superstring*, *JHEP* **04** (2000) 018 [[hep-th/0001035](#)].
- [2] E. Cartan, *Leçon sur la Théorie des Spineurs*, Hermann, Paris, 1937; *The theory of spinors*, MIT Press, 1966.
- [3] N. Berkovits, *Cohomology in the pure spinor formalism for the superstring*, *JHEP* **09** (2000) 046 [[hep-th/0006003](#)].
- [4] N. Berkovits and B.C. Vallilo, *Consistency of super-Poincaré covariant superstring tree amplitudes*, *JHEP* **07** (2000) 015 [[hep-th/0004171](#)].
- [5] N. Berkovits, *Multiloop amplitudes and vanishing theorems using the pure spinor formalism for the superstring*, *JHEP* **09** (2004) 047 [[hep-th/0406055](#)].
- [6] N. Berkovits, *Super-Poincaré covariant two-loop superstring amplitudes*, *JHEP* **01** (2006) 005 [[hep-th/0503197](#)];  
N. Berkovits and C.R. Mafra, *Equivalence of two-loop superstring amplitudes in the pure spinor and RNS formalisms*, *Phys. Rev. Lett.* **96** (2006) 011602 [[hep-th/0509234](#)];  
C.R. Mafra, *Four-point one-loop amplitude computation in the pure spinor formalism*, *JHEP* **01** (2006) 075 [[hep-th/0512052](#)].
- [7] D. Friedan, E.J. Martinec and S.H. Shenker, *Conformal invariance, supersymmetry and string theory*, *Nucl. Phys.* **B 271** (1986) 93.
- [8] E. D'Hoker and D.H. Phong, *Two-loop superstrings, I. Main formulas*, *Phys. Lett.* **B 529** (2002) 241 [[hep-th/0110247](#)]; *Two-loop superstrings, II. The chiral measure on moduli space*, *Nucl. Phys.* **B 636** (2002) 3 [[hep-th/0110283](#)]; *Two-loop superstrings, III. Slice independence and absence of ambiguities*, *Nucl. Phys.* **B 636** (2002) 61 [[hep-th/0111016](#)]; *Two-loop superstrings, IV. The cosmological constant and modular forms*, *Nucl. Phys.* **B 639** (2002) 129 [[hep-th/0111040](#)]; *Two-loop superstrings, V. Gauge slice independence of the N-point function*, *Nucl. Phys.* **B 715** (2005) 91 [[hep-th/0501196](#)]; *Two-loop superstrings, VI. Non-renormalization theorems and the 4-point function*, *Nucl. Phys.* **B 715** (2005) 3 [[hep-th/0501197](#)].
- [9] N. Berkovits and O. Chandia, *Superstring vertex operators in an  $AdS_5 \times S^5$  background*, *Nucl. Phys.* **B 596** (2001) 185 [[hep-th/0009168](#)];  
B.C. Vallilo, *One loop conformal invariance of the superstring in an  $AdS_5 \times S^5$  background*, *JHEP* **12** (2002) 042 [[hep-th/0210064](#)]; *One loop conformal invariance of the superstring in an  $AdS_5 \times S^5$  background*, *JHEP* **12** (2002) 042 [[hep-th/0210064](#)].
- [10] N. Berkovits, *Pure spinor formalism as an  $N = 2$  topological string*, *JHEP* **10** (2005) 089 [[hep-th/0509120](#)].
- [11] N.A. Nekrasov, *Lectures on curved beta-gamma systems, pure spinors and anomalies*, [hep-th/0511008](#).
- [12] E. Witten, *Topological sigma models*, *Commun. Math. Phys.* **118** (1988) 411;  
M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Holomorphic anomalies in topological field theories*, *Nucl. Phys.* **B 405** (1993) 279 [[hep-th/9302103](#)].
- [13] A. Gaona and J.A. Garcia, *BFT embedding of the Green-Schwarz superstring and the pure spinor formalism*, *JHEP* **09** (2005) 083 [[hep-th/0507076](#)].

- [14] N. Berkovits, *Covariant quantization of the superstring*, *Int. J. Mod. Phys. A* **16** (2001) 801 [[hep-th/0008145](#)]; *Relating the RNS and pure spinor formalisms for the superstring*, *JHEP* **08** (2001) 026 [[hep-th/0104247](#)]; *Covariant quantization of the superparticle using pure spinors*, *JHEP* **09** (2001) 016 [[hep-th/0105050](#)];  
 N. Berkovits and O. Chandia, *Lorentz invariance of the pure spinor BRST cohomology for the superstring*, *Phys. Lett. B* **514** (2001) 394 [[hep-th/0105149](#)];  
 N. Berkovits and P.S. Howe, *Ten-dimensional supergravity constraints from the pure spinor formalism for the superstring*, *Nucl. Phys. B* **635** (2002) 75 [[hep-th/0112160](#)];  
 N. Berkovits, *Conformal field theory for the superstring in a Ramond-Ramond plane wave background*, *JHEP* **04** (2002) 037 [[hep-th/0203248](#)];  
 N. Berkovits and O. Chandia, *Massive superstring vertex operator in  $D = 10$  superspace*, *JHEP* **08** (2002) 040 [[hep-th/0204121](#)];  
 N. Berkovits and V. Pershin, *Supersymmetric Born-Infeld from the pure spinor formalism of the open superstring*, *JHEP* **01** (2003) 023 [[hep-th/0205154](#)].
- [15] B.C. Vallilo, *Flat currents in the classical  $AdS_5 \times S^5$  pure spinor superstring*, *JHEP* **03** (2004) 037 [[hep-th/0307018](#)];  
 N. Berkovits, *BRST cohomology and nonlocal conserved charges*, *JHEP* **02** (2005) 060 [[hep-th/0409159](#)]; *Quantum consistency of the superstring in  $AdS_5 \times S^5$  background*, *JHEP* **03** (2005) 041 [[hep-th/0411170](#)].
- [16] N. Berkovits, *Covariant quantization of the supermembrane*, *JHEP* **09** (2002) 051 [[hep-th/0201151](#)].
- [17] N. Berkovits and D.Z. Marchioro, *Relating the Green-Schwarz and pure spinor formalisms for the superstring*, *JHEP* **01** (2005) 018 [[hep-th/0412198](#)].
- [18] P.A. Grassi, G. Policastro, M. Porrati and P. Van Nieuwenhuizen, *Covariant quantization of superstrings without pure spinor constraints*, *JHEP* **10** (2002) 054 [[hep-th/0112162](#)];  
 P.A. Grassi, G. Policastro and P. van Nieuwenhuizen, *The massless spectrum of covariant superstrings*, *JHEP* **11** (2002) 001 [[hep-th/0202123](#)]; *On the BRST cohomology of superstrings with/without pure spinors*, *Adv. Theor. Math. Phys.* **7** (2003) 499 [[hep-th/0206216](#)]; *The covariant quantum superstring and superparticle from their classical actions*, *Phys. Lett. B* **553** (2003) 96 [[hep-th/0209026](#)]; *Yang-Mills theory as an illustration of the covariant quantization of superstrings*, [hep-th/0211095](#);  
 P.A. Grassi, G. Policastro and P. van Nieuwenhuizen, *An introduction to the covariant quantization of superstrings*, *Class. Quant. Grav.* **20** (2003) S395–S410 [[hep-th/0302147](#)];  
*The quantum superstring as a WZNW model*, *Nucl. Phys. B* **676** (2004) 43 [[hep-th/0307056](#)]; *Superstrings and WZNW models*, [hep-th/0402122](#).
- [19] I. Oda and M. Tonin, *On the Berkovits covariant quantization of GS superstring*, *Phys. Lett. B* **520** (2001) 398 [[hep-th/0109051](#)]; *The B-field in pure spinor quantization of superstrings*, [hep-th/0510223](#);  
 M. Matone, L. Mazzucato, I. Oda, D. Sorokin and M. Tonin, *The superembedding origin of the Berkovits pure spinor covariant quantization of superstrings*, *Nucl. Phys. B* **639** (2002) 182 [[hep-th/0206104](#)];  
 I. Oda, *Covariant matrix model of superparticle in the pure spinor formalism*, *Mod. Phys. Lett. A* **18** (2003) 1023 [[hep-th/0302203](#)].
- [20] G. Trivedi, *Correlation functions in Berkovits' pure spinor formulation*, *Mod. Phys. Lett. A* **17** (2002) 2239 [[hep-th/0205217](#)].

- [21] Y. Aisaka and Y. Kazama, *A new first class algebra, homological perturbation and extension of pure spinor formalism for superstring*, *JHEP* **02** (2003) 017 [[hep-th/0212316](#)]; *Operator mapping between RNS and extended pure spinor formalisms for superstring*, *JHEP* **08** (2003) 047 [[hep-th/0305221](#)]; *Relating Green-Schwarz and extended pure spinor formalisms by similarity transformation*, *JHEP* **04** (2004) 070 [[hep-th/0404141](#)].
- [22] Y. Aisaka and Y. Kazama, *Origin of pure spinor superstring*, *JHEP* **05** (2005) 046 [[hep-th/0502208](#)].
- [23] M. Chesterman, *Ghost constraints and the covariant quantization of the superparticle in ten dimensions*, *JHEP* **02** (2004) 011 [[hep-th/0212261](#)]; *On the cohomology and inner products of the Berkovits superparticle and superstring*, *Nucl. Phys. B* **703** (2004) 400 [[hep-th/0404021](#)].
- [24] R. Schiappa and N. Wyllard, *D-brane boundary states in the pure spinor superstring*, *JHEP* **07** (2005) 070 [[hep-th/0503123](#)];  
P. Mukhopadhyay, *On D-brane boundary state analysis in pure-spinor formalism*, *JHEP* **03** (2006) 066 [[hep-th/0505157](#)];  
P. Mukhopadhyay, *DDF construction and D-brane boundary states in pure spinor formalism*, [hep-th/0512161](#).
- [25] T. Oota, *Comments on equations of motion for pure spinors in even dimensions*, [hep-th/411036](#).
- [26] S. Guttenberg, J. Knapp and M. Kreuzer, *On the covariant quantization of type-II superstrings*, *JHEP* **06** (2004) 030 [[hep-th/0405007](#)].
- [27] P.A. Grassi and L. Tamassia, *Vertex operators for closed superstrings*, *JHEP* **07** (2004) 071 [[hep-th/0405072](#)];  
P.A. Grassi and P. Vanhove, *Topological M-theory from pure spinor formalism*, [hep-th/0411167](#);  
L. Anguelova, P.A. Grassi and P. Vanhove, *Covariant one-loop amplitudes in  $D = 11$* , *Nucl. Phys. B* **702** (2004) 269 [[hep-th/0408171](#)];  
P.A. Grassi and J.F. Morales Morera, *Partition functions of pure spinors*, [hep-th/0510215](#).
- [28] P.A. Grassi and N. Wyllard, *Lower-dimensional pure-spinor superstrings*, *JHEP* **12** (2005) 007 [[hep-th/0509140](#)];  
N. Wyllard, *Pure-spinor superstrings in  $D = 2, 4, 6$* , *JHEP* **11** (2005) 009 [[hep-th/0509165](#)];  
O. Chandia,  *$D = 4$  pure spinor superstring and  $N = 2$  strings*, *JHEP* **10** (2005) 105 [[hep-th/0509185](#)].
- [29] N. Berkovits, *ICTP lectures on covariant quantization of the superstring*, [hep-th/0209059](#).
- [30] W. Siegel, *Hidden local supersymmetry in the supersymmetric particle action*, *Phys. Lett. B* **128** (1983) 397.
- [31] E. Bergshoeff, E. Sezgin and P.K. Townsend, *Supermembranes and eleven-dimensional supergravity*, *Phys. Lett. B* **189** (1987) 75.
- [32] L. Brink and J.H. Schwarz, *Quantum superspace*, *Phys. Lett. B* **100** (1981) 310.
- [33] E. Bergshoeff, E. Sezgin and Y. Tani, *Hamiltonian formulation of the supermembrane*, *Nucl. Phys. B* **298** (1988) 187.
- [34] R. Arnowitt, S. Deser and C.W. Misner, *The dynamics of general relativity*, Wiley, New York, 1962, chap. 7.

- [35] M. Henneaux and C. Teitelboim, *Quantization of gauge systems*, Princeton University Press, 1992.
- [36] S. Carlip, *Heterotic string path integrals with the Green-Schwarz covariant action*, *Nucl. Phys. B* **284** (1987) 365;  
R. Kallosh and A.Y. Morozov, *Green-Schwarz action and loop calculations for superstring*, *Int. J. Mod. Phys. A* **3** (1988) 1943.